Pairs of non-homogeneous linear differential polynomials

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Abstract

In [4], Langley proved a result concerning the zeros of pairs of (possibly non-homogeneous) linear differential polynomials in a meromorphic function. We generalise this result by relaxing Langley's assumption on the frequency of zeros (counting multiplicity), and present some results based on restricting the order of the differential operators.

Introduction

Let f be a non-constant meromorphic function in the plane, and let $S(r,f)$ denote any quantity which is $o(T(r,f))$ as $r \to \infty$ outside a set of finite measure, using the standard notation. Throughout we use the convention that c_s is a "small function" - i.e. a function such that $T(r,c_s) = S(r,f)$. We further define a linear differential polynomial ψ in f by

$$\psi = \sum_{s=0}^{t} c_s f^{(s)}. \tag{1}$$

We begin with a result of Milloux from [3], which may be viewed as a counterpart of Nevanlinna's Second Fundamental Theorem.

Theorem (Milloux's result)

Let f be meromorphic and non-constant in the plane, and ψ as defined by (1) also be non-constant. Then,

$$T(r,f) < \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{\psi-1}\right) - N_0\left(r,\frac{1}{\psi'}\right) + S(r,f), \quad (2)$$

where $N_0(r,1/\psi')$ counts only zeros of ψ' which are not multiple 1-points of ψ.

Hayman showed that for $\psi = f^{(k)}$ with $k \geq 1$, a version of (2) holds without the $\overline{N}(r,f)$ term, and in particular that if f omits 0 then $f^{(k)}$ must take every finite non-zero value. This was subsequently extended by Bergweiler and Langley in [1] to linear differential polynomials in f, subject to conditions on the coefficients c_j. We in turn consider pairs of such polynomials.

Definitions

- L and M are linear differential operators, of order k and n respectively, with coefficients which are rational functions, where the equations $L(\omega) = 0$ and $M(\omega) = 0$ have no common non-trivial (local) solutions.
- P and Q are linear differential operators such that $P(L) + Q(M) = 1$.
- $F = L(f) + a$, $G = M(f) + b$ where a and b are rational functions.
- $g = P(F) + Q(G) = f + c$, where $c = P(a) + Q(b)$.
- Ω is a non-empty simply-connected domain on which the functions a, b, and the coefficients in L and M are analytic.
- u and v are solutions to $L(\omega) = a$ and $M(\omega) = b$ respectively, $u_1, \ldots, u_k$ and $v_1, \ldots, v_n$ are linearly independent solutions of $L(\omega) = 0$ and $M(\omega) = 0$ respectively, all well-defined on Ω.

Langley's result [4]

Let the function f be transcendental and meromorphic in the plane, and let the definitions hold. Assume that

$$N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{G}\right) = S(r,f). \tag{3}$$

Then at least one of the following holds:

1. *$F = L(g)$ and $G = M(g)$;*
2. *f has a representation $f = R(u_1,\ldots,u_k,v_1,\ldots,v_n,u,v)$, where R is a rational function in $k+n+2$ variables.*

The main theorem

Let the function f be transcendental and meromorphic in the plane, and let the definitions hold. Assume that $\gamma_1 \geq 0$, $\gamma_2 \geq 0$ and that

$$N\left(r,\frac{1}{F}\right) \leq \gamma_1 T(r,f) + S(r,f), \qquad N\left(r,\frac{1}{G}\right) \leq \gamma_2 T(r,f) + S(r,f). \tag{4}$$

Further define

$$\gamma_0 = \max\{\gamma_1,\gamma_2\}, \qquad \gamma_3 = \gamma_1 + \gamma_2. \tag{5}$$

Then at least one of the following holds:

1. *$F = L(g)$ and $G = M(g)$;*
2. *f has a representation $f = R(u_1,\ldots,u_k,v_1,\ldots,v_n,u,v)$, where R is a rational function in $k+n+2$ variables;*
3. *the γ_j satisfy*

$$\frac{k+n}{k+n+1} \leq 2\gamma_3 + \gamma_0 + \frac{2\gamma_3+1}{k+n}. \tag{6}$$

A theorem for the case $k = n$

Let f be transcendental and meromorphic in the plane, let the definitions hold with $k = n$, and assume that

$$\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) = S(r,f). \tag{7}$$

Then at least one of the following must hold:

1. *$F = L(g)$ and $G = M(g)$;*
2. *f has a representation $f = R(u_1,\ldots,u_n,v_1,\ldots,v_n,u,v)$, where R is a rational function in $2n+2$ variables.*

A theorem for the case $n > k+2$

Let f be transcendental and meromorphic in the plane, and let the definitions hold with $b \equiv 0$ and $a - L(c) \not\equiv 0$. Suppose that

$$\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) = S(r,f), \tag{8}$$

and further that

$$n > k+2. \tag{9}$$

Then at least one of the following holds:

1. *$F = L(g)$ and $G = M(g)$;*
2. *f has a representation $f = R(u_1,\ldots,u_k,v_1,\ldots,v_n,u,v)$, where R is a rational function in $k+n+2$ variables.*

Remarks

- The proof of the main theorem follows the method of [4], and we note here that setting $\gamma_1 = \gamma_2 = 0$ leads to a contradiction in (6) and thus returns Langley's original result. It is also clear that the inequality will hold for any k and n if $\gamma_3 \geq 0.4$.
- The second theorem shows that if $k = n$ then N can be replaced by $\overline{N}$ in the hypothesis (3) of Langley's result. This theorem uses a significantly different proof method to the main theorem, based on the quotient F/G.
- These results are due to appear in the March 2011 edition of the Computational Methods and Function Theory journal (*www.cmft.de*).

References

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