Pairs of Non-Homogeneous Linear Differential Polynomials

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Abstract

In [8], Langley proved a result concerning the zeros of pairs of (possibly non-homogeneous) linear differential polynomials in a meromorphic function. We generalise this result by relaxing Langley's assumption on the frequency of zeros (counting multiplicity), and further prove some results based on restricting the order of the differential operators.

Contents

1 Introduction

Let f be a non-constant meromorphic function in the plane, and let $S(r, f)$ denote any quantity which is $o(T(r, f))$ as $r \to \infty$ outside a set of finite measure. Throughout we use the standard notation from [6]. Throughout this section, we will use the convention that c_s is a "small function" - i.e. a function such that $T(r, c_s) = S(r, f)$. We further define a linear differential polynomial ψ in f by

$$\psi = \sum_{s=0}^{t} c_s f^{(s)}. \tag{1.1}$$

We begin with several results of Milloux from [6].

Proposition 1.1

Let f be meromorphic and non-constant in the plane, and ψ as defined by (1.1) also be non-constant. Then,

$$T(r, f) < \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right) + S(r, f), \tag{1.2}$$

where $N_0(r, 1/\psi')$ counts only zeros of ψ' which are not multiple 1-points of ψ.

Corollary 1.2

Let f be meromorphic and transcendental in the plane, with only a finite number of zeros and poles. Then every function ψ, as defined in (1.1), assumes every finite complex value, except possibly 0*, infinitely often, or else is identically constant.*

Hayman in [6] used Milloux's results in the case $\psi = f^{(t)}$ to obtain what is referred to as "Hayman's Alternative".

Proposition 1.3 - Hayman's Alternative

Let f be transcendental and meromorphic in the plane. Then either f assumes every finite value infinitely often, or $f^{(t)}$ assumes every finite value except possibly 0 *infinitely often for every positive integer t.*

We can refine this result in the following way, as noted by Nevo, Pang and Zalcman in [11].

Corollary 1.4

Let f be a meromorphic function in the plane. If $f \neq 0$ and $f^{(t)} \neq 1$ for some fixed positive integer t, then f is constant.

Examples 1.5

(i) Let $f = e^z$. Since $f \neq 0$, all derivatives of f, which are in fact equal to f, take every finite non-zero value infinitely often.

(ii) The trigonometric functions $\sin z$ and $\cos z$ take every finite value infinitely often. This is clear since here $f^{(4)} = f$, and we know that f takes the value 0 infinitely often.

(iii) Since we know that $\tan z \neq \pm i$, we now know that all derivatives of $\tan z$ take all finite non-zero values infinitely often. The same is true for derivatives of the functions $\csc z$, and $\sec z$, neither of which take the value 0 in the plane, and $\cot z$, which also fails to take the values $\pm i$.

We state a result by Frank, Hennekemper, Langley and Polloczek [1][2][7] which confirmed a conjecture of Hayman [6].

Proposition 1.6 [4]

Suppose that f is meromorphic in the plane and that f and $f^{(k)}$ have only finitely many zeros, for some $k \geq 2$. Then we have $f = Re^P$ where R is a rational function in z and P a polynomial in z. In particular, f has finite order and finitely many poles.

Frank and Langley in [5] proved a result for homogeneous linear differential operators.

Proposition 1.7

Let f be meromorphic and non-constant in the plane, and let L_1 and L_2 be homogeneous linear differential operators, with coefficients which are rational functions and leading terms $\dfrac{d^k}{dz^k}$ and $\dfrac{d^n}{dz^n}$ respectively, with $k \geq n \geq 1$. Let $F = L_1(f)$ and $G = L_2(f)$, assume that

$$\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) = O\left(\log^+ T\left(r, \frac{f'}{f}\right) + \log r\right)$$

as $r \to \infty$ outside a set of finite measure and further assume that the equations $L_1(\omega) = 0$ and $L_2(\omega) = 0$ have no non-trivial common (local) solutions, so that in particular L_1 and L_2 are not the same.

Then f has finite order and finitely many zeros and f'/f has a representation

$$\frac{f'[z]}{f[z]} = Y[z] + \frac{P\left[Q + \log R[z]\right]\left(Q'[z] + R'[z]/R[z]\right)}{R[z]e^{Q[z]} - 1}$$

in which Y and R are rational functions and P and Q are polynomials, and at least one of P and R is constant.

We now make some definitions which will be used throughout the rest of this paper.

Definitions 1.8

Let L and M be linear differential operators of positive order k and n respectively, *with*

$$L=\frac{d^k}{dz^k}+\sum_{j=0}^{k-1}a_j\frac{d^j}{dz^j},\qquad M=\frac{d^n}{dz^n}+\sum_{j=0}^{n-1}b_j\frac{d^j}{dz^j}, \tag{1.3}$$

where the coefficients a_j, b_j are rational functions, and where the equations $L(\omega)=0$ and $M(\omega)=0$ have no common non-trivial (local) solutions. Then by lemmas from [5], there exist linear differential operators P, Q, U, V and Y with coefficients which are rational functions in the a_j, b_j and their derivatives such that

$$P(L)+Q(M)=1,\qquad Y=U(L)=V(M), \tag{1.4}$$

where 1 is the identity operator, and U, V, Y, have order n, k, $n+k$, and leading terms $\frac{d^n}{dz^n}$, $\frac{d^k}{dz^k}$, $\frac{d^{n+k}}{dz^{n+k}}$ respectively. The (local) solution space of $Y(\omega)=0$ is the direct sum of the (local) solution spaces of the equations $M(\omega)=0$ and $L(\omega)=0$. The parentheses in (1.4) denote composition. We now define linear differential polynomials F and G by

$$F=L(f)+a,\qquad G=M(f)+b, \tag{1.5}$$

where f is meromorphic in the plane, and a, b are rational functions and we assume that $F\not\equiv 0$, $G\not\equiv 0$. We define a rational function c by

$$c=P(a)+Q(b), \tag{1.6}$$

and set

$$g=f+c=P(F)+Q(G). \tag{1.7}$$

Now, from these definitions, we see that

$$F=L(f)+a=L(g)+a-L(c),\qquad G=M(f)+b=M(g)+b-M(c). \tag{1.8}$$

Furthermore,

$$U(F)=V(G)+d,\qquad d=U(a)-V(b). \tag{1.9}$$

where d is a rational function.

Finally, let Ω be a non-empty simply-connected domain on which the functions a, b, and the coefficients a_j, b_j are analytic. Further define on Ω linearly independent solutions $u_1, \dots, u_k$ of $L(\omega) = 0$, linearly independent solutions $v_1, \dots, v_n$ of $M(\omega) = 0$, and solutions u and v of $L(\omega) = a$ and $M(\omega) = b$ respectively.

We now state Langley's result from [8], which provides our springboard for the results which follow.

Proposition 1.9

Let the function f be transcendental and meromorphic in the plane, and suppose that Definitions 1.8 hold. Assume that

$$N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) = S(r, f). \tag{1.10}$$

Then at least one of the following holds:

1. $F = L(g)$ *and* $G = M(g)$*;*
2. f *has a representation* $f = R(u_1, \dots, u_k, v_1, \dots, v_n, u, v)$*, where* R *is a rational function in* $k + n + 2$ *variables.*

We now give an example from [8].

Example 1.10

Let $f = e^z + e^{2z} + 1$*, and*

$$F(z) = f'(z) - f(z) + 1 = e^{2z}, \qquad G(z) = f'(z) - 2f(z) + 2 = -e^z.$$

This satisfies both conditions 1 and 2.

2 The results

Our first result weakens the assumption (1.10).

Theorem 2.1

Let the function f be transcendental and meromorphic in the plane, and let Definitions 1.8 hold. Assume that $\gamma_1 \geq 0$, $\gamma_2 \geq 0$ and that

$$N\left(r, \frac{1}{F}\right) \leq \gamma_1 T(r,f) + S(r,f), \qquad N\left(r, \frac{1}{G}\right) \leq \gamma_2 T(r,f) + S(r,f). \tag{2.1}$$

Further define

$$\gamma_0 = \max\{\gamma_1, \gamma_2\}, \qquad \gamma_3 = \gamma_1 + \gamma_2. \tag{2.2}$$

Then at least one of the following holds:

1. $F = L(g)$ *and* $G = M(g)$*;*

2. f *has a representation* $f = R(u_1, \ldots, u_k, v_1, \ldots, v_n, u, v)$*, where* R *is a rational function in* $k + n + 2$ *variables.*

3. *the* γ_j *satisfy*

$$1 \leq \frac{2\gamma_3(k+n+1)+1}{k+n+1} + \frac{2\gamma_3+1}{k+n} + \gamma_0 \tag{2.3}$$

Remark: We note here that conclusions 1 and 2 are as in Proposition 1.9, and that setting $\gamma_1 = \gamma_2 = 0$ leads to an immediate contradiction in (2.3), and thus returns Proposition 1.9.

Theorem 2.2

Let f be transcendental and meromorphic in the plane, let Definitions 1.8 hold with $k = n$, and assume that

$$\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) = S(r,f). \tag{2.4}$$

Then at least one of the following must hold:

1. *$F = L(g)$ and $G = M(g)$;*

2. *f has a representation $f = R(u_1, \ldots, u_n, v_1, \ldots, v_n, u, v)$, where R is a rational function in $2n + 2$ variables.*

Thus Theorem 2.2 shows that if $k = n$ then N can be replaced by $\overline{N}$ in the hypothesis (1.10) of Proposition 1.9.

Theorem 2.3

Let f be transcendental and meromorphic in the plane, and let Definitions 1.8 hold with $b \equiv 0$ and $a - L(c) \not\equiv 0$. Suppose that (2.4) holds and suppose further that

$$n > k + 2. \tag{2.5}$$

Then at least one of the following holds:

1. *$F = L(g)$ and $G = M(g)$;*

2. *f has a representation $f = R(u_1, \ldots, u_k, v_1, \ldots, v_n, u, v)$, where R is a rational function in $k + n + 2$ variables.*

Theorem 2.4

Let f be transcendental and meromorphic in the plane, and let Definitions 1.8 and (1.10) hold.

If $a - L(c) \not\equiv 0$, then at least one of the following holds:

1. *f has finitely many poles;*

2. $n \leq 2$;

3. *f has a rational representation $f = R(f_1, \ldots, f_{n+1})$ where the f_j are (local) solutions to $M(\omega) = d_j b$ and each d_j is a constant.*

If $b - M(c) \not\equiv 0$, then at least one of the following holds:

1. *f has finitely many poles;*

2. *$k \leq 2$;*

3. *f has a rational representation $f = R(f_1, \ldots, f_{k+1})$ where the f_j are (local) solutions to $L(\omega) = d_j a$ and each d_j is a constant.*

3 Preliminary lemmas

In this section we state and then refine a lemma from [8] which will be very useful in our proofs.

Lemma 3.1 [8]

Let δ be a positive real number, and let the function h be transcendental and meromorphic in the plane. Let p be a positive integer, and $c_0, c_1, \ldots, c_{p-1}$ and A be rational functions. Set

$$Q_p = \frac{d^p}{dz^p} + \sum_{j=0}^{p-1} c_j \frac{d^j}{dz^j},$$
$$H = Q_p(h) + A.$$

Then at least one of the following conditions holds:

(i) we have, as $r \to \infty$,

$$p\overline{N}(r,h) \leq N\left(r, \frac{1}{H}\right) + (1+\delta)N(r,h) + S(r,h); \tag{3.1}$$

(ii) h has a representation

$$h = R(h_1, \ldots, h_{p+1}), \tag{3.2}$$

where R is a rational function in $p+1$ variables and each h_j is a (local) solution of

$$Q_p(\omega) = d_j A, \tag{3.3}$$

with d_j a constant.

We now present a refinement of Langley's result.

Lemma 3.2

Let h in Lemma 3.1 be such that (ii) does not hold. Then:

$$p\overline{N}(r,h) \leq N\left(r, \frac{1}{H}\right) + N(r,h) + S(r,h). \tag{3.4}$$

Proof:

Assuming that Lemma 3.1 (ii) does not hold, then (i) must hold for any $\delta > 0$. In particular, for all $n \in \mathbb{N}$,

$$\begin{aligned} p\overline{N}(r,h) &\leq N\left(r, \frac{1}{H}\right) + \left(1 + \frac{1}{2n}\right) N(r,h) + S(r,h) \\ &\leq N\left(r, \frac{1}{H}\right) + N(r,h) + \frac{1}{n}T(r,h) \end{aligned}$$

for all $r \geq 1$ outside a set E_n of finite measure. Now, take a sequence (r_n) such that $F_n = E_n \cap [r_n, \infty)$ has measure at most n^{-2}, with $r_n \geq r_{n-1} + 1$, and $r_1 \geq 1$. Then $r_n \to \infty$. Let

$$F_0 = \bigcup_{n=1}^{\infty} F_n.$$

Then F_0 has measure at most $1 + 2^{-2} + 3^{-2} + \ldots < \infty$. Now let $r \notin F_0$ be large, and r_m the largest member of (r_n) which is not greater than r. Then m is large and $r \in [r_m, \infty)$. However, $r \notin E_m$, so

$$p\overline{N}(r,h) \leq N\left(r, \frac{1}{H}\right) + N(r,h) + \frac{1}{m}T(r,h).$$

Thus, as $r \to \infty$ with $r \notin F_0$ (and thus $m \to \infty$),

$$p\overline{N}(r,h) \leq N\left(r, \frac{1}{H}\right) + N(r,h) + o(T(r,h)),$$

which leads immediately to (3.4) by our definition of $S(r,h)$.

QED

4 Proof

4.1 Initial steps

Assume that f is transcendental meromorphic in the plane, and that the Definitions 1.8 hold. We state and prove several lemmas.

Lemma 4.1

If either $U(F)$ or $V(G)$ is a rational function then f has finitely many poles and at least one of the following must hold:

1. *The following inequality holds:*

$$T(r,f) \leq N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+S(r,f) \tag{4.1}$$

$$\leq n\overline{N}\left(r,\frac{1}{F}\right)+k\overline{N}\left(r,\frac{1}{G}\right)+S(r,f) \tag{4.2}$$

2. *f has a representation $f = R(u_1,\ldots,u_k,v_1,\ldots,v_n,u,v)$, where R is a rational function in $k+n+2$ variables, and the u_j, v_j, u and v are as in Definitions 1.8.*

Proof:

Assume without loss of generality that $U(F)$ is rational; then by (1.9) so is $V(G)$. Assume that neither vanishes identically. Then F and G each solve a non-homogeneous linear differential equation with rational coefficients, and

$$\frac{U(F)}{F}=\frac{F^{(n)}}{F}+\ldots=\frac{R_0}{F},$$

where $R_0 \not\equiv 0$ is a rational function. Thus, by the Lemma of the Logarithmic Derivative [6],

$$m\left(r,\frac{1}{F}\right) \leq m\left(r,\frac{U(F)}{F}\right)+O(\log r)=S(r,f),$$

and similarly for G. Thus, by the First Fundamental Theorem,

$$\begin{aligned} T(r,F)+T(r,G) &\leq T\left(r,\frac{1}{F}\right)+T\left(r,\frac{1}{G}\right)+S(r,f) \\ &\leq N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+S(r,f). \end{aligned} \tag{4.3}$$

Now, we have that

$$P(F)=\sum_{j=0}^{p}\alpha_j F^{(j)},$$

where α_j are rational coefficients, and thus, by the Lemma of the Logarithmic Derivative,

$$m(r,P(F))\leq m(r,F)+m\left(r,\frac{P(F)}{F}\right)+S(r,f)=m(r,F)+S(r,f).$$

Moreover, as $U(F)$ is rational, F has only finitely many poles, and so does f. Thus,

$$N(r,P(F))=O(\log r),$$

and similarly for $Q(G)$. Thus, by (1.7),

$$\begin{aligned} T(r,f) &\leq T(r,g)+S(r,f) \\ &\leq T(r,F)+T(r,G)+S(r,f), \end{aligned}$$

to which we apply (4.3) to obtain (4.1). Now, a zero z_0 of F of multiplicity $m>n$ with z_0 large is a zero of $U(F)$ of multiplicity at least $m-n$, but this is impossible since $U(F)$ is rational. Thus

$$N\left(r,\frac{1}{F}\right)\leq n\overline{N}\left(r,\frac{1}{F}\right)+S(r,f),$$

and similarly for G. We then apply this to (4.1) to obtain (4.2).

Now assume without loss of generality that $U(F)\equiv 0$. Then by (1.4) and (1.5), $0=U(F)=Y(f)+U(a)$ so that with the u_j, v_j and u as defined, $f+u$ solves $Y(\omega)=0$ and is a linear combination of $u_1,\ldots,u_k,v_1,\ldots,v_n$ on Ω. Thus f has a representation as asserted.

QED

It is clear from (1.8) that if $a - L(c)$ and $b - M(c)$ both vanish identically, then $F = L(g)$ and $G = M(g)$ are satisfied. Hence we assume in the next lemma that at least one of $B = a - L(c)$ and $C = b - M(c)$ does not vanish identically.

Lemma 4.2

Assume that both $U(F)$ and $V(G)$ are transcendental. Then if $a - L(c) \not\equiv 0$

$$T(r,f) \le N\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f), \tag{4.4}$$

and

$$T(r,f) \le (k+1)\overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f). \tag{4.5}$$

If $b - M(c) \not\equiv 0$ then

$$T(r,f) \le N\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{G}\right) + S(r,f), \tag{4.6}$$

and

$$T(r,f) \le (n+1)\overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{G}\right) + S(r,f). \tag{4.7}$$

Proof:

Assume that $B = a - L(c) \not\equiv 0$. We take $f + c = g = Bg^*$ and $F = L(g) + B = B(L^*(g^*) + 1)$ where B is a rational function and L^* is a linear differential operator, and hence

$$\overline{N}\left(r,\frac{1}{L^*(g^*)+1}\right) = \overline{N}\left(r,\frac{1}{F}\right) + S(r,f), \tag{4.8}$$

and

$$T(r,f) = T(r,g^*) + S(r,f), \quad \overline{N}(r,f) = \overline{N}(r,g) + S(r,f) = \overline{N}(r,g^*) + S(r,f). \tag{4.9}$$

Since $U(F)$ is transcendental, so is F, and thus $(L^*(g^*)+1)' \not\equiv 0$, and so we may apply Milloux's result (1.2) to g^*, giving

$$T(r,g^*) \le \overline{N}(r,g^*) + N\left(r,\frac{1}{g^*}\right) + \overline{N}\left(r,\frac{1}{L^*(g^*)+1}\right) - N_0\left(r,\frac{1}{(L^*(g^*))'}\right) + S(r,g^*), \tag{4.10}$$

where N_0 counts the zeros of $(L^*(g^*))'$ which are not also zeros of $L^*(g^*)+1$. Now, a zero z_0 of g^* of multiplicity p with z_0 large contributes p to $n(r,1/g^*)$ and at least $\max\{0,p-k-1\}$ to $n_0(r,1/(L^*(g^*))')$, and hence at most $\min\{p,k+1\}\le k+1$ to $n(r,1/g^*)-n_0(r,1/(L^*(g^*))')$. Hence, we can rewrite (4.10) as

$$T(r,g^*)\le \overline{N}(r,g^*)+(k+1)\overline{N}\left(r,\frac{1}{g^*}\right)+\overline{N}\left(r,\frac{1}{L^*(g^*)+1}\right)+S(r,g^*),$$

and thus, by application of (4.8) and (4.9), we obtain (4.5). Now, returning to (4.10), we may discard the N_0 term and apply (4.9) to give (4.4).

We follow similar steps to obtain (4.6) and (4.7).

QED

Lemma 4.3

Assume that $U(F)$ and $V(G)$ are transcendental. If $d=U(F)-V(G)\not\equiv 0$, then

$$N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{H}\right)\le \overline{N}(r,f)+2N\left(r,\frac{1}{F}\right)+2N\left(r,\frac{1}{G}\right)+S(r,f), \qquad (4.11)$$

and

$$\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{H}\right)\le \overline{N}(r,f)+A_1\left(\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\right)+S(r,f), \quad (4.12)$$

where A_1 is a positive constant and

$$H=\left(\frac{d}{dz}-\frac{d'}{d}\right)(U(F))\not\equiv 0. \qquad (4.13)$$

If $d\equiv 0$, then

$$N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{H}\right)\le 2N\left(r,\frac{1}{F}\right)+2N\left(r,\frac{1}{G}\right)+S(r,f), \qquad (4.14)$$

and

$$\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{H}\right)\le A_2\left(\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\right)+S(r,f), \qquad (4.15)$$

where A_2 *is a positive constant and*

$$H = U(F) \not\equiv 0. \tag{4.16}$$

Proof:

Assume first that $d = U(F) - V(G) \not\equiv 0$. We define linear differential operators $\widetilde{U}$ and $\widetilde{V}$ by

$$\widetilde{U} = \left(\frac{d}{dz} - \frac{d'}{d}\right)(U), \qquad \widetilde{V} = \left(\frac{d}{dz} - \frac{d'}{d}\right)(V), \tag{4.17}$$

and thus by (4.13) and the definition of d, we have $H = \widetilde{U}(F) = \widetilde{V}(G)$. If $H \equiv 0$, then by the previous two equations, there exist constants μ, ν such that $U(F) = \mu d$ and $V(G) = \nu d$, which, since d is rational by (1.9), contradicts our assumption that $U(F)$ and $V(G)$ are transcendental. Thus, $H \not\equiv 0$. Set

$$\phi = \frac{gH}{FG} = \frac{P(F)\widetilde{V}(G)}{FG} + \frac{Q(G)\widetilde{U}(F)}{FG}, \tag{4.18}$$

using (1.7) and (4.13). Since $P, Q, \widetilde{U}$ and $\widetilde{V}$ are linear differential operators with rational functions as coefficients, by the Lemma of the Logarithmic Derivative [6]

$$m(r, \phi) = S(r, f).$$

We now turn to $N(r, \phi)$. Suppose f has a pole of multiplicity m at some point z_0 with z_0 large. Then g, F, G and H have poles at z_0 with multiplicities $m, m+k, m+n$ and $m+n+k+1$ respectively, and so ϕ has a simple pole at z_0. Thus,

$$\begin{aligned} T(r,\phi) &\leq N(r,\phi) + S(r,f) \\ &\leq \overline{N}(r,f) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + S(r,f). \end{aligned} \tag{4.19}$$

Writing $1/gH = 1/\phi FG$ and using (4.19) leads to, since g has only finitely many poles at zeros of H (and vice versa),

$$\begin{aligned}N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{H}\right) &\leq N\left(r,\frac{1}{gH}\right)+S(r,f) \qquad (4.20)\\ &= N\left(r,\frac{1}{\phi FG}\right)+S(r,f)\\ &\leq N\left(r,\frac{1}{\phi}\right)+N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+S(r,f)\\ &\leq T(r,\phi)+N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+S(r,f), \qquad (4.21)\end{aligned}$$

from which (4.11) follows from substitution of (4.19).

Since a zero of F (respectively G) gives at most a pole of $F^{(j)}/F$ (respectively $G^{(j)}/G$) of multiplicity j, we obtain, for some $\widetilde{A_1}>0$,

$$T(r,\phi)\leq \overline{N}(r,f)+\widetilde{A_1}\left(\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\right)+S(r,f). \qquad (4.22)$$

Again, as g has only finitely many poles at zeros of H (and vice versa),

$$\begin{aligned}\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{H}\right) &\leq \overline{N}\left(r,\frac{1}{gH}\right)+S(r,f) \qquad (4.23)\\ &= \overline{N}\left(r,\frac{1}{\phi FG}\right)+S(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{\phi}\right)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)\\ &\leq T(r,\phi)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f), \qquad (4.24)\end{aligned}$$

from which (4.12) follows from substitution of (4.22).

Now consider the case where $d\equiv 0$. Then we define H and ϕ using (1.7) and (1.9) by

$$H=U(F)=V(G), \qquad \phi=\frac{gH}{FG}=\frac{P(F)V(G)}{FG}+\frac{Q(G)U(F)}{FG}, \qquad (4.25)$$

and $H\not\equiv 0$ by our assumption that $U(F)$ and $V(G)$ are transcendental. Again, $m(r,\phi)=S(r,f)$, but here the only poles of ϕ are due to zeros of FG, as the poles of f cancel each other out. Thus, (4.19) becomes

$$T(r,\phi)\leq \widetilde{A_2}\left(N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)\right)+S(r,f), \qquad (4.26)$$

which when substituted into (4.21) yields (4.14). (4.26) and (4.24) then yield (4.15).

QED

4.2 Proof of Theorem 2.1

Assume the hypotheses of the theorem. Suppose first that at least one of $U(F)$ and $V(G)$ is rational. Then by Lemma 4.1, either conclusion 2 of the theorem holds, or

$$\begin{aligned} T(r,f) &\leq N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+S(r,f) \\ &\leq (\gamma_1+\gamma_2)T(r,f)+S(r,f), \end{aligned}$$

and hence $\gamma_3=\gamma_1+\gamma_2\geq 1$ which implies (2.3). We henceforth assume that both $U(F)$ and $V(G)$ are transcendental. Furthermore, if $a-L(c)\equiv b-M(c)\equiv 0$, then we have conclusion 1 of the theorem by (1.8). We henceforth assume that this is not the case. Then by Lemma 4.2,

$$T(r,f)\leq N\left(r,\frac{1}{g}\right)+\overline{N}(r,f)+\gamma_0 T(r,f)+S(r,f), \tag{4.27}$$

where $\gamma_0=\max\{\gamma_1,\gamma_2\}$. We now divide the proof into two cases.

<u>Case I</u>

Suppose that $d=U(F)-V(G)\not\equiv 0$ in (1.9). Then by Lemma 4.3, we have

$$\begin{aligned} N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{H}\right) &\leq \overline{N}(r,f)+2N\left(r,\frac{1}{F}\right)+2N\left(r,\frac{1}{G}\right)+S(r,f) \\ &\leq \overline{N}(r,f)+2\gamma_3 T(r,f)+S(r,f), \end{aligned} \tag{4.28}$$

where

$$H=(U(F))'-\frac{d'}{d}U(F)\not\equiv 0.$$

Now, by (1.4) and (1.5), H has a representation

$$H=\left(\left(\frac{d}{dz}-\frac{d'}{d}\right)(Y)\right)(f)+\left(\left(\frac{d}{dz}-\frac{d'}{d}\right)(U)\right)(a)$$

as a (possibly non-homogeneous) linear differential polynomial in f, of order $k+n+1$, with rational functions as coefficients. Lemmas 3.1 and 3.2 now give two possibilities, one of which is that f has a representation

$$f = R(y_1, \ldots, y_{k+n+2}),$$

where R is a rational function in $k+n+2$ variables and each y_j is a (local) solution of

$$\left(\left(\frac{d}{dz} - \frac{d'}{d}\right)(Y)\right)(\omega) = d_j \left(\left(\frac{d}{dz} - \frac{d'}{d}\right)(U)\right)(a)$$

where each d_j is a constant. But, for some constant e_j, using (1.9),

$$Y(y_j) = d_j U(a) + e_j d = (d_j + e_j)U(a) - e_j V(b).$$

Hence, with u_j, v_j, u and v as defined, $y - (d_j + e_j)u + e_j v$ solves $Y(\omega) = 0$ on Ω and is a linear combination of $u_1, \ldots, u_k, v_1, \ldots, v_n$, and so conclusion 2 of the theorem is satisfied. The other possibility is that

$$(k+n+1)\overline{N}(r,f) \le N\left(r, \frac{1}{H}\right) + N(r,f) + S(r,f). \tag{4.29}$$

We combine this with (4.28), yielding

$$\begin{aligned} N\left(r, \frac{1}{g}\right) + (k+n+1)\overline{N}(r,f) &\le N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{H}\right) + N(r,f) + S(r,f) \\ &\le \overline{N}(r,f) + (2\gamma_3 + 1)T(r,f) + S(r,f), \end{aligned}$$

and so

$$N\left(r, \frac{1}{g}\right) + (k+n)\overline{N}(r,f) \le (2\gamma_3 + 1)T(r,f) + S(r,f),$$

which leads to

$$\overline{N}(r,f) \le \frac{2\gamma_3 + 1}{k+n} T(r,f) - \frac{1}{k+n} N\left(r, \frac{1}{g}\right) + S(r,f). \tag{4.30}$$

We add $N\left(r, \frac{1}{g}\right)$ to both sides and use (4.28), yielding

$$\frac{k+n+1}{k+n} N\left(r, \frac{1}{g}\right) \le \left(2\gamma_3 + \frac{2\gamma_3 + 1}{k+n}\right) T(r,f) + S(r,f).$$

We then substitute this inequality and (4.30) into (4.27) to give

$$T(r,f) \leq \left(\frac{k+n}{k+n+1} \left(2\gamma_3 + \frac{2\gamma_3+1}{k+n} \right) + \frac{2\gamma_3+1}{k+n} + \gamma_0 \right) T(r,f) + S(r,f),$$

from which (2.3) is immediate.

<u>Case II</u>

Now suppose that $d \equiv 0$. Then by Lemma 4.3 we have

$$\begin{aligned} N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{H}\right) &\leq 2N\left(r, \frac{1}{F}\right) + 2N\left(r, \frac{1}{G}\right) + S(r,f) \\ &\leq 2\gamma_3 T(r,f) + S(r,f), \end{aligned} \tag{4.31}$$

where $H = U(F)$. Now, by (1.4) and (1.5), H has a representation

$$H = Y(f) + U(a)$$

as a (possibly non-homogeneous) linear differential polynomial in f, of order $k+n$, with rational functions as coefficients. Lemmas 3.1 and 3.2 now give two possibilities, one of which is that f has a representation

$$f = R(y_1, \ldots, y_{k+n+1}),$$

where R is a rational function in $k+n+1$ variables and each y_j is a (local) solution of

$$Y(\omega) = d_j U(a)$$

where each d_j is a constant, so that $y_j - d_j u$ solves $Y(\omega) = 0$, and thus conclusion 2 of the theorem is satisfied. The other possibility is that

$$(k+n)\overline{N}(r,f) \leq N\left(r, \frac{1}{H}\right) + N(r,f) + S(r,f).$$

Substituting in (4.31), we obtain

$$\overline{N}(r,f) \leq \frac{2\gamma_3+1}{k+n} T(r,f) + S(r,f). \tag{4.32}$$

We now substitute (4.31) into (4.27) to give

$$(1-2\gamma_3-\gamma_0)T(r,f) \leq \overline{N}(r,f) + S(r,f).$$

We substitute this into (4.32) and thus obtain

$$(1-2\gamma_3-\gamma_0)T(r,f) \leq \frac{2\gamma_3+1}{k+n}T(r,f) + S(r,f),$$

which is a stronger condition than (2.3).

QED

4.3 Proof of Theorem 2.2

Assume the hypotheses of the theorem. If either $U(F)$ or $V(G)$ is rational, then by Lemma 4.1, either conclusion 2 of Theorem 2.2 holds, or by (2.4) and (4.2) we have $T(r,f) = S(r,f)$, which is a contradiction. Henceforth assume that both $U(F)$ and $V(G)$ are transcendental. Then if $a - L(c) \equiv b - M(c) \equiv 0$ then by (1.8) conclusion 1 of Theorem 2.2 holds. Assume henceforth without loss of generality that $a - L(c) \not\equiv 0$. Then by Lemmas 4.2 and 4.3 and (2.4),

$$\begin{aligned} T(r,f) &\leq (n+1)\overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,f) + S(r,f) \\ &\leq (n+2)\overline{N}(r,f) + S(r,f), \end{aligned}$$

and in particular,

$$\overline{N}(r,f) \neq S(r,f). \tag{4.33}$$

We define $\lambda_j = \psi^{(j)}/\psi$ for $j \geq 0$, and in particular, $\lambda = \lambda_1 = \psi'/\psi$, where

$$\psi = \frac{L(f)+a}{M(f)+b} = \frac{F}{G}.$$

If $\psi \equiv c_1$ for some constant c_1, and thus $\lambda \equiv 0$, then

$$L(f) + a = F = c_1 G = c_1(M(f)+b).$$

Since the equations $L(\omega) = 0$ and $M(\omega) = 0$ have no non-trivial solutions in common, f solves a (possibly non-homogeneous) linear differential equation with rational coefficients, and f has only finitely many poles, contradicting (4.33). Thus $\lambda \not\equiv 0$. Since F and G have the same order, all but finitely many poles of f are 1-points of ψ by (1.3) and (1.5). Now, by (2.4),

$$\begin{aligned} N(r,\lambda) &= \overline{N}\left(r,\frac{1}{\psi}\right) + \overline{N}\left(r,\psi\right) && (4.34)\\ &\leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + O(\log r) && (4.35)\\ &= S(r,f). && (4.36)\end{aligned}$$

Since λ is the quotient of a function and its derivative, by the Lemma of the Logarithmic Derivative [6], we have $m(r,\lambda) = S(r,f)$, and so

$$T(r,\lambda) = S(r,f). \qquad (4.37)$$

Now, for $j \in \mathbb{N}$,

$$\begin{aligned} F^{(j)} = (\psi G)^{(j)} &= \sum_{\phi=0}^{j} \binom{j}{\phi} \psi^{(j-\phi)} G^{(\phi)} && (4.38)\\ &= \psi \sum_{\phi=0}^{j} \binom{j}{\phi} \lambda_{j-\phi} G^{(\phi)}. && (4.39)\end{aligned}$$

Let U and V be as in (1.4), and write

$$U(F) = \sum_{j=0}^{n} p_j F^{(j)} \qquad V(G) = \sum_{j=0}^{n} q_j G^{(j)},$$

with $p_n = q_n = 1$. Then (1.9) gives

$$\begin{aligned} V(G) + d &= U(F) \\ &= \sum_{j=0}^{n} p_j F^{(j)} \\ &= \psi \sum_{j=0}^{n} p_j \sum_{\phi=0}^{j} \binom{j}{\phi} \lambda_{j-\phi} G^{(\phi)} \\ &= \psi \sum_{j=0}^{n} p_j \sum_{\phi=0}^{n} \binom{j}{\phi} \lambda_{j-\phi} G^{(\phi)} \\ &= \psi \sum_{\phi=0}^{n} \sum_{j=0}^{n} p_j \binom{j}{\phi} \lambda_{j-\phi} G^{(\phi)} \\ &= \psi \sum_{\phi=0}^{n} \sum_{j=\phi}^{n} p_j \binom{j}{\phi} \lambda_{j-\phi} G^{(\phi)} \qquad (4.40) \end{aligned}$$

using the property that $\binom{s}{t} = 0$ for $t \notin [0, s]$. Now, suppose that z_0 is large and a pole of f of order m. Then z_0 is a pole of G of order $m+n$, and for $0 \leq l \leq n$, $X_l = G^{(l)}/G^{(n)}$ has a zero of order $n - l$ at z_0. Further, we have $\psi(z_0) = 1$ and $\lambda_l(z_0) \in \mathbb{C}\ \forall l \geq 0$. Thus at z_0, by the definition of $V(G)$,

$$\frac{V(G) + d}{G^{(n)}} = 1 + q_{n-1} X_{n-1} + M_1 \qquad (4.41)$$

where M_j will denote functions with a zero of multiplicity at least two at z_0. But we also have, using (1.9) and (4.40),

$$\frac{V(G) + d}{G^{(n)}} = \frac{U(F)}{G^{(n)}} = \psi \left(1 + (p_{n-1} + n\lambda) X_{n-1} + M_2\right). \qquad (4.42)$$

Now, G has a pole of multiplicity $m + n$ at z_0, and so $G^{(j)}$ has a pole of multiplicity $m + n + j$, and we may write, as $z \to z_0$,

$$X_{n-1} \sim \frac{(z - z_0)^{-(m+2n-1)}}{-(m + 2n - 1)(z - z_0)^{-(m+2n)}} = \frac{-(z - z_0)}{m + 2n - 1}. \qquad (4.43)$$

We then differentiate, giving $X'_{n-1}(z_0) = -(m + 2n - 1)^{-1}$. We have using (4.41) and (4.42),

$$1 + q_{n-1} X_{n-1} + M_1 = \psi \left(1 + (p_{n-1} + n\lambda) X_{n-1} + M_2\right), \qquad (4.44)$$

which we now differentiate to give, setting $\widetilde{p} = p_{n-1} + n\lambda$ and using $\lambda = \psi'/\psi$,

$$q'_{n-1}X_{n-1} + q_{n-1}X'_{n-1} + M'_1 = \lambda\psi\left(1 + \widetilde{p}X_{n-1} + M_2\right) + \psi\left(\widetilde{p}X'_{n-1} + \widetilde{p}'X_{n-1} + M'_2\right). \tag{4.45}$$

We now substitute in $\psi(z_0) = 1$, $X_{n-1}(z_0) = 0$, $X'_{n-1}(z_0) = -(m+2n-1)^{-1}$ and $M_j(z_0) = M'_j(z_0) = 0$, then rearrange to give, at z_0,

$$\lambda - \frac{n\lambda}{m+2n-1} = \frac{p_{n-1} - q_{n-1}}{m+2n-1} \tag{4.46}$$

$$\lambda = \frac{p_{n-1} - q_{n-1}}{m+n-1}. \tag{4.47}$$

Now, suppose there exists no $m \in \mathbb{N}$ such that $\lambda \equiv \dfrac{p_{n-1} - q_{n-1}}{m+n-1}$. We define $N^m(r, f)$ to be the counting function $N(r, f)$ restricted to those poles of multiplicity m. We rearrange (4.47) to give

$$\Lambda = \lambda - \frac{p_{n-1} - q_{n-1}}{m+n-1} = 0, \tag{4.48}$$

and so

$$\overline{N}^m(r, f) \leq N\left(r, \frac{1}{\Lambda}\right) + S(r, f), \tag{4.49}$$

where the $S(r, f)$ term takes account of the finitely many poles excluded from our analysis above. But

$$T(r, \Lambda) \leq T(r, \lambda) + T(r, p_{n-1}) + T(r, q_{n-1}) + S(r, f), \tag{4.50}$$

which, by (4.37) and the fact that the p_j and q_j are rational functions, is itself $S(r, f)$. Thus, since $\Lambda \not\equiv 0$ by assumption,

$$\overline{N}^m(r, f) \leq T(r, \Lambda) + S(r, f) = S(r, f). \tag{4.51}$$

Now, let $\varepsilon > 0$. For each $\kappa \in \mathbb{N}$ we may choose r_κ such that

$$\sum_{m=1}^{\kappa} \overline{N}^m(r, f) = S(r, f) = o(T(r, f)) \leq \frac{1}{\kappa} T(r, f) \tag{4.52}$$

for $r \geq r_\kappa$ outside some set E_κ of measure at most κ^{-2}. We further assume that $r_{\kappa+1} \geq r_\kappa$ $\forall \kappa \in \mathbb{N}$. Then $E = \bigcup_{\kappa\in\mathbb{N}} E_\kappa$ has finite measure. Let κ be big enough that $2\kappa^{-1} \leq \varepsilon$, and r large and not in E. Then $r \geq r_\kappa$, $r \notin E_\kappa$, so

$$\sum_{m\leq\kappa} \overline{N}^m(r,f) \leq \frac{1}{\kappa}T(r,f) \leq \frac{\varepsilon}{2}T(r,f) \tag{4.53}$$

$$\sum_{m>\kappa} \overline{N}^m(r,f) \leq \frac{1}{\kappa}N(r,f) \leq \frac{1}{\kappa}T(r,f) \leq \frac{\varepsilon}{2}T(r,f), \tag{4.54}$$

and so

$$\overline{N}(r,f) \leq \varepsilon T(r,f). \tag{4.55}$$

This holds for all sufficiently large $r \notin E$, and so

$$\overline{N}(r,f) = S(r,f), \tag{4.56}$$

which contradicts (4.33).

It follows that there is some $m \in \mathbb{N}$ such that

$$\lambda \equiv \frac{p_{n-1} - q_{n-1}}{m+n-1}, \tag{4.57}$$

and so λ is a rational function. But, since $\psi' = \lambda\psi$, this means that

$$\psi = \frac{F}{G} = Re^P \tag{4.58}$$

for some rational function R and polynomial P, and thus ψ has only finitely many poles and zeros. Suppose that F has infinitely many zeros of large multiplicity. Then all but finitely many of these are zeros of G of the same multiplicity, since ψ has finitely many zeros and poles, and they are zeros of $U(F)$ and $V(G)$, and thus of d by (1.9). Thus, there are infinitely many zeros of d, and so, since d is rational, $d \equiv 0$.

We now follow a slight variation on our previous argument. Suppose z_0 is large and a zero of G of multiplicity $\mu \geq n$. Then $X_j = G^{(j)}/G^{(n)}$ has a zero of multiplicity $n-j$ at z_0 for $0 \leq j \leq n$. As before, $X_{n-1}(z_0) = 0$, but now, as $z \to z_0$,

$$X_{n-1}(z) \sim \frac{(z-z_0)^{\mu-n+1}}{(\mu-n+1)(z-z_0)^{\mu-n}}, \tag{4.59}$$

and so $X'_{n-1}(z_0) = (\mu - n + 1)^{-1}$. We again have (4.41) and (4.42), and hence (4.44) and (4.45). We also have that $\psi(z_0) = 1$ by (4.41) and (4.42). We substitute this and (4.59) into (4.45), giving

$$\frac{q_{n-1}}{\mu - n + 1} = \lambda + \frac{p_{n-1} + n\lambda}{\mu - n + 1}, \tag{4.60}$$

which we rearrange to find

$$\lambda = \frac{q_{n-1} - p_{n-1}}{\mu + 1}. \tag{4.61}$$

However, we already have an identity for λ by (4.57), which we equate to give

$$\frac{q_{n-1} - p_{n-1}}{\mu + 1} = \lambda \equiv \frac{p_{n-1} - q_{n-1}}{m + n - 1} \tag{4.62}$$

$$\frac{-1}{\mu + 1} = \frac{1}{m + n - 1}. \tag{4.63}$$

But, $\mu > 0$ and $m + n > 1$, and so we are trying to equate one number which is strictly positive and another which is strictly negative - clearly impossible. Thus, our supposition that there are infinitely many such zeros of large multiplicity is false, and so there are only finitely many of these zeros, which thus contribute $O(\log r) = S(r, f)$ to $N(r, 1/F)$ and $N(r, 1/G)$. Hence, by (2.4),

$$N\left(r, \frac{1}{F}\right) \leq n\overline{N}\left(r, \frac{1}{F}\right) + S(r, f) = S(r, f), \tag{4.64}$$

$$N\left(r, \frac{1}{G}\right) \leq n\overline{N}\left(r, \frac{1}{G}\right) + S(r, f) = S(r, f), \tag{4.65}$$

and thus we can apply Langley's Proposition 1.9 with $k = n$ and the conclusions of Theorem 2.2 follow immediately.

QED

4.4 Proof of Theorem 2.3

We begin by stating a refinement of Lemma 3.1 from [9].

Lemma 4.4.1 [9]

Let $0 < \varepsilon < 1$, and let L_0 be a homogeneous linear differential operator of order $p \geq 2$ with rational functions for coefficients. Let h be transcendental and meromorphic in the plane, and $H = L_0(h)$. Then at least one of the following two conclusions holds:

1. *h is rational in p (local) solutions of $L_0(\omega) = 0$;*

2. *the functions h and H satisfy*

$$N(r, H) \leq C\overline{N}\left(r, \frac{1}{H}\right) + (2 + \varepsilon)N(r, h) + S(r, f) \tag{4.66}$$

where $C \leq 2(1 + \varepsilon)^{p\lambda}$ for any $1 < \lambda \in \mathbb{R}$ satisfying

$$\log(1 + \varepsilon) \geq \frac{\log(1 + \lambda)}{\lambda} + \log\left(1 + \frac{1}{\lambda}\right).$$

We note here that H is required to be homogeneous. We may now begin the proof.

Assume the hypotheses of Theorem 2.3. By Lemma 4.1, if either $U(F)$ or $V(G)$ are rational then either conclusion 2 of our theorem holds, or (4.2) holds, which, by (2.4), means that $T(r, f) \leq S(r, f)$, which is a contradiction. Assume henceforth that $U(F)$ and $V(G)$ are transcendental. By Lemma 4.3 and (2.4),

$$\overline{N}\left(r, \frac{1}{g}\right) \leq \overline{N}(r, f) + S(r, f).$$

We combine this with (4.5) of Lemma 4.2, which holds under our assumption that $a - L(c) \not\equiv 0$, to yield

$$N(r, f) \leq T(r, f) \leq (k + 2)\overline{N}(r, f) + S(r, f). \tag{4.67}$$

We apply Lemma 4.4.1 with $p = k$, $h = f$, $H = G = M(f)$ and ε small. If conclusion 1 of the lemma holds, then f is a rational function in solutions of $M(\omega) = 0$, and so conclusion 2 of the theorem holds. Assume this is not the case. Then (4.66) holds, and we apply (2.4), and thus obtain

$$N(r, G) \leq (2 + \varepsilon)N(r, f) + S(r, f) \tag{4.68}$$

Now, the poles of G are caused either by poles of the coefficients, which since they are rational contribute $S(r,f)$, or by poles of f, where if z_0 is large and a pole of f of order m, then it is a pole of G of order $m+n$. Thus,

$$N(r,G) = N(r,f) + n\overline{N}(r,f) + S(r,f),$$

which we substitute into (4.68), giving

$$n\overline{N}(r,f) \leq (1+\varepsilon)N(r,f) + S(r,f),$$

and hence by substitution of (4.67),

$$n\overline{N}(r,f) \leq (1+\varepsilon)(k+2)\overline{N}(r,f) + S(r,f).$$

By (4.67) we have $\overline{N}(r,f) \neq S(r,f)$, and so $n \leq (1+\varepsilon)(k+2)$. Since ε may be chosen arbitrarily small, this contradicts (2.5).

QED

4.5 Proof of Theorem 2.4

We present the proof for $a - L(c) \not\equiv 0$; the proof when $b - M(c) \not\equiv 0$ follows along the same lines.

Suppose that one of $U(F)$ or $V(G)$ is rational. Then, as noted in Lemma 4.1, f has finitely many poles. Now, suppose that $U(F)$ and $V(G)$ are transcendental. Then by (4.4) and Lemma 4.3,

$$T(r,f) \leq 2\overline{N}(r,f) + 2N\left(r,\frac{1}{F}\right) + 2N\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f),$$

which, by (1.10) reduces to

$$T(r,f) \leq 2\overline{N}(r,f) + S(r,f), \tag{4.69}$$

and in particular $\overline{N}(r,f) \neq S(r,f)$. Applying Lemma 3.2 with $p = n$, $Q_p = M$ and $H = G$ gives either conclusion 3 of the theorem, or, by substitution of (4.69) and (1.10),

$$\begin{aligned} n\overline{N}(r,f) &\leq N\left(r, \frac{1}{G}\right) + N(r,f) + S(r,f) \\ &\leq 2\overline{N}(r,f) + S(r,f), \end{aligned}$$

from which conclusion 2 is immediate.

QED

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