Integer points of meromorphic functions

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Integer-valued functions

A function which takes integer values on some subset E of $\mathbb{Z}$.

Examples:

$$\begin{aligned}
&f(z) = 5z^6 + 9z^2 + 1 && E = \mathbb{Z} \\
&f(z) = \sin(\pi z) && E = \mathbb{Z} \\
&f(z) = n^z \text{ for } n \in \mathbb{Z} && E = \mathbb{N} \cup \{0\} \\
&f(z) = \Gamma(z) && E = \mathbb{N} \\
&f(z) = \frac{z}{n} \text{ for } n \in \mathbb{Z} \setminus \{0\} && E = n\mathbb{Z}
\end{aligned}$$

Pólya's original result

Theorem (Pólya, c1915)

Let f *be an entire function such that* $f(n) \in \mathbb{Z}$ *for* $n = 0, 1, 2, \ldots$ *and*

$$\limsup_{r \to \infty} \frac{\log M(r, f)}{r} < \log 2.$$

Then f *is a polynomial.*

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We cannot replace $\limsup$ here by $\liminf$:

Theorem (Langley, 2006)

Given $\psi(r) \to \infty$, there exists a transcendental entire function G with $G(\mathbb{Z}) \subseteq \mathbb{Z}$ and

$$\liminf_{r\to\infty} \frac{\log M(r, G)}{\psi(r)\log r} < \infty.$$

A corollary and a conjecture

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2^z is the slowest-growing transcendental entire function mapping $\{0, 1, 2, \ldots\}$ into $\mathbb{Z}$.

What happens if we specify what integer values are taken?

Conjecture (Buck/Langley, 2011)

For $n \in \mathbb{N}$, 2^{nz} is the slowest-growing transcendental entire function mapping $\{0, 1, 2, \ldots\}$ into the set of n^{th} powers of $\mathbb{Z}$.

Further results

Theorem (Pólya, 1921)

Let f be an entire function such that $f(n) \in \mathbb{Z}$ for $n = 0, 1, 2, \ldots$ and

$$M(r, f) = O\left(r^m 2^r\right) \quad \text{as} \quad r \to \infty$$

for some $m > 0$. Then

$$f(z) = P_1(z)2^z + P_2(z) \tag{1}$$

where the P_j are polynomials.

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Theorem (Pisot, 1942)

Let f be an entire function such that $f(n) \in \mathbb{Z}$ for $n = 0, 1, 2, \ldots$. Then (1) holds if

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{r} \lessapprox 0.7589.$$

A half-plane result

Theorem (Fletcher/Langley, 2009)

Let $d \in (0,1)$, $J \in \mathbb{N}$ and $\lambda > 0$ satisfy

$$\frac{16}{J}\left(1 + \log\left(\frac{J}{2} + 1\right)\right) + 8(J-1)\lambda < d^2,$$

and let $E \subseteq \mathbb{N}$ have lower linear density

$$\underline{D}(E) = \liminf_{n\to\infty} \frac{|E \cap \{1,\ldots,n\}|}{n} > d.$$

Further let f be an analytic function of exponential type less than λ in the closed right half plane, and assume that $f(n) \in \mathbb{Z}$ for every $n \in E$. Then f is a polynomial.

This gives a maximal value for λ of roughly $\dfrac{1}{2800} \approx 3.6 \times 10^{-4}$.

A generalisation of Fletcher/Langley to meromorphic functions in the whole plane

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Theorem (Buck, 2011)

Given $d \in (0,1)$, there exists some $\lambda = \lambda(d) > 0$ with the following property. Let f be meromorphic in the plane, taking integer values on some set $E \subseteq \mathbb{N}$ of positive lower linear density $d_0 > d$, and $T(r,f) \leq \lambda r$ for all $r \geq r_0$. Then f is a polynomial.

Outline proof

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- Use the Boutroux-Cartan Lemma to give a bound for $\log|f(z)|$ within a disc of large radius R, excepting a union U of open discs which enclose at most a third of the points of E within the large disc. Call this region Δ, and enumerate the elements of $\Delta \cap E$ as $\alpha_1 < \alpha_2 < \dots$.

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- Using the Dirichlet Box Principle, we can create a finite sum

$$A(z) = \sum_{\mu,\nu} \beta_{\mu,\nu} \binom{z}{z-\mu} f(z)^{\nu},$$

where the $\beta_{\mu,\nu}$ are integers, not all 0, and $|A(\alpha_j)| < 1$. Then $A(E) \subseteq \mathbb{Z}$, and so $A(\alpha_j) = 0$.

Outline proof

- Use Boutroux-Cartan to show that given these zeros of A in the interval $[1, R]$, if $T(r, A) \leq \vartheta r$ for all $r \geq R$, then it has more zeros within $[1, 2R]$. Repeat infinitely.

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- If $T(r, A)$ is small enough and $A \not\equiv 0$ then

$$N\left(r, \frac{1}{A}\right) > (1 + o(1))\, T(r, A),$$

contradicting the First Fundamental Theorem. Hence,

$$A(z) \equiv 0 \equiv \sum_{\nu} \left(\sum_{\mu} \beta_{\mu,\nu} \left(\frac{z}{z - \mu} \right) \right) f(z)^{\nu}.$$

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- Apply a lemma of Fletcher/Langley says that if an algebraic function f is analytic on the right half plane and integer-valued on E, then f is a polynomial.

How small is λ?

Short answer: very.

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λ must satisfy the following inequality:

$$\frac{4}{J}\left(1+\log\left(\frac{J}{2}+1\right)\right)+8(J-1)\left(6-\frac{\log\frac{d}{96}}{\log 2}\right)\lambda < \frac{d\log\frac{7}{6}}{16\left(6-\frac{\log\frac{d}{96}}{\log 2}\right)},$$

where $J \in \mathbb{N}$ is large enough that

$$J > \frac{128\log\frac{d}{6144}}{d\log\frac{7}{6}\log 2} W\left(\frac{d\log\frac{7}{6}\log 2\exp\left(\frac{d\log\frac{7}{6}\log 2}{64\log\frac{d}{6144}}-1\right)}{64\log\frac{d}{6144}}\right)-2,$$

where W is the Lambert W-function.

How small is λ?

Solving for specific values of d gives the following results for J and λ:

$$
\begin{aligned}
d &= 1 & J &\gtrapprox 130,000 & \lambda &\lessapprox 2.9 \times 10^{-11} \approx \frac{1}{3.4 \times 10^{10}} \\
d &= 0.5 & J &\gtrapprox 290,000 & \lambda &\lessapprox 5.6 \times 10^{-12} \approx \frac{1}{1.8 \times 10^{11}} \\
d &= 0.1 & J &\gtrapprox 2,000,000 & \lambda &\lessapprox 1.2 \times 10^{-13} \approx \frac{1}{8.3 \times 10^{12}} \\
d &= 0.01 & J &\gtrapprox 28,000,000 & \lambda &\lessapprox 5.8 \times 10^{-16} \approx \frac{1}{1.7 \times 10^{15}}
\end{aligned}
$$

Further avenues of investigation

- Can we change the set E to be a subset of $\mathbb{Z}$ but not of $\mathbb{N}$?
- Can this result for meromorphic functions be used in a half plane?
- Can the conjecture on the slowest-growing transcendental entire function mapping to n^{th}powers of $\mathbb{Z}$ be proven?