Integer points of meromorphic functions

Matthew Buck [1]

Abstract

Working from a half-plane result of Fletcher and Langley [2], we show that if f is an integer-valued function on some subset of the natural numbers of positive lower density and is meromorphic of sufficiently small exponential type in the plane, then f is a polynomial.

A.M.S. Classification: 30D30, 30D35
Keywords: Integer-valued functions; Meromorphic functions; Nevanlinna Theory; Value distribution.

1 Introduction

An integer-valued function is one such that $f(\mathbb{Z}) \subseteq \mathbb{Z}$, a simple example being a polynomial with integer coefficients, or $\sin(\pi z)$. Research in this field generally focusses on functions which are integer-valued on some subset of $\mathbb{Z}$. Pólya proved an early result in this field.

Proposition 1 [7]
Let f be entire, taking integer values on $\mathbb{N} \cup \{0\}$, and suppose that

$$\limsup_{r\to\infty} \frac{M(r,f)}{2^r} < 1$$

where $M(r,f)$ is the maximum modulus function of f. Then f is a polynomial.

Langley in [5] later showed that the lim sup cannot be replaced by a lim inf. A corollary to Pólya's result is that 2^z is the slowest growing transcendental entire function to take integer values on the non-negative integers. Curiously, no research appears to have been done into what happens if we further restrict the values which may be taken at the integer points, to say prime numbers, or square numbers. Pólya further showed that

Proposition 2 [7]

[1] This research was conducted as part of the author's PhD thesis studies, and was supported by a grant from the Engineering and Physical Sciences Research Council.

Let f be an entire function such that $f(n) \in \mathbb{Z}$ for $n = 0, 1, 2, \ldots$ and

$$\limsup_{r \to \infty} \frac{\log M(r)}{r} \leq \alpha \leq \log 2.$$

Then there exist polynomials $P_j(z)$ such that

$$f(z) = P_1(z)2^z + P_2(z).$$

This was later improved to $\alpha \leq \log 2 + \frac{1}{1500}$ by Selberg in [9], then further by Pisot in [6].

Fletcher and Langley proved a half-plane analogue to Pólya's result [2],

Proposition 3 [2]
Let d, J and λ satisfy

$$0 < d < 1, \quad J \in \mathbb{N}, \quad \lambda > 0, \quad \frac{16}{J}\left(1 + \log\left(1 + \frac{J}{2}\right)\right) + 8(J-1)\lambda < d^2.$$

Let $E \subset \mathbb{N}$ have lower density

$$\underline{D}(E) = \liminf_{n \to \infty} \frac{|E \cap \{1, \ldots, n\}|}{n} > d,$$

let f be analytic of exponential type less than λ in the closed right half plane, and assume that $f(n) \in \mathbb{Z}$ for every $n \in E$. Then f is a polynomial.

Further results on integer-valued entire functions may be found in [1], [8] and [11], among others. However, there does not appear to have been any research into whether an analogue of Pólya's result can be obtained for meromorphic functions. In this paper, we generalise Fletcher and Langley's result to meromorphic functions, following the general method of their proof, which was in turn based on a method of Waldschmidt [10]. However, our result is restricted to functions which are meromorphic in the whole plane rather than a half plane.

2 The theorem

Given $d \in (0, 1)$, there exists some $\lambda = \lambda(d) > 0$ with the following property. Let f be meromorphic in the plane, taking integer values on some set $E \subseteq \mathbb{N}$ of lower density $d_0 > d$, with $T(r, f) \leq \lambda r$ for all $r \geq r_0$. Then f is a polynomial.

We will calculate how small λ needs to be in the appendix.

3 Lemmata

We begin with some lemmas. The first is an elementary result comparing the integrated and unintegrated counting functions.

Lemma 1
Let $0 < s < S$, and let h be a meromorphic function on the set $|x| \leq S$. Then

$$N(S,h) \geq n(s,h) \log \frac{S}{s} + n(0,h) \log s.$$

This is a well-known result, and so we omit the proof. The next lemma is found in many texts, including [3], where it is presented as a mass distribution result. A more elementary proof can be found in [4].

Lemma 2 - The (Boutroux-)Cartan Lemma
Let $z_1, \ldots, z_n \in \mathbb{C}$, and $\gamma > 0$. Then

$$V(z) = \sum_{j=1}^{n} \log|z - z_j| > n \log \gamma \tag{3.1}$$

for all z outside a union U of open discs of total radius at most 6γ.

Remark: We may assume that the discs are disjoint, since if some point z_0 is within two discs, of radius r_1 and r_2 respectively, we may choose a new disc of radius $r_3 < r_1 + r_2$ that encloses both original discs. We may also assume that each disc contains at least one z_j, as otherwise (3.1) applies on the boundary of that disc, and since V is harmonic inside the disc, we may extend (3.1) to the interior. We may therefore assume that there are at most n discs.

We now apply Boutroux-Cartan to give a bound on the logarithm of the modulus of a function in terms of its Nevanlinna characteristic.

Lemma 3
Let $m \geq 0$, $s \geq 1$, $0 < \varepsilon \leq 1$, and let h be meromorphic on the set $|z| \leq 8s$ with at least m distinct zeros in $|z| \leq s$. Then

$$\log|h(z)| \leq \left(6 - \frac{\log \varepsilon}{\log 2}\right) T(8s,h) + m \log \frac{6}{7} \tag{3.2}$$

for all $|z| \leq 2s$ lying outside a union U of at most $n(4s,h)$ open discs of total radius at most $24\varepsilon s$.

Remark: A disc of radius $s > 0$ contains at most $1 + 2s$ distinct integers, and so the number of integers in U is at most the number of discs plus double the total radius.

Proof:

Let $S = 4s$ and $n = n(4s, h)$, and further let $b_1, \ldots, b_n$ be the poles of h in $|z| \leq S$, repeated according to multiplicity. If $m > 0$, let $a_1, \ldots, a_m$ be distinct zeros of h in $|z| \leq s$. Finally, define the function g by

$$g(z) = h(z) \prod_{j=1}^{m} \frac{S^2 - \overline{a_j} z}{S(z - a_j)} \prod_{k=1}^{n} \frac{z - b_k}{S}$$

where an empty product is taken as 1. Thus g is analytic on $|z| \leq S$. Also, for $|z| = S$, we have

$$\left| \frac{S^2 - \overline{a_j} z}{S(z - a_j)} \right| = 1 \quad \text{and} \quad \left| \frac{z - b_k}{S} \right| \leq 2,$$

and so,

$$T(S, g) = m(S, g) \leq m(S, h) + n(S, h) \log 2.$$

Since $S > 1$, we have by Lemma 1

$$N(2S, h) \geq n(S, h) \log 2,$$

and so

$$T(S, g) \leq m(S, h) + N(2S, h) \leq 2T(2S, h).$$

Thus, by the standard comparison between the maximum modulus and characteristic functions for functions analytic on a disc centred at the origin, we have for $|z| \leq 2s = S/2$,

$$\log |g(z)| \leq \frac{S + \frac{S}{2}}{S - \frac{S}{2}} T(S, g) = 3T(S, g) \leq 6T(2S, h) = 6T(8s, h).$$

Also in this region we have $|z - a_j| \leq 3s$ and $|S^2 - \overline{a_j} z| \geq 14s^2$, and so

$$\left| \frac{S(z - a_j)}{S^2 - \overline{a_j} z} \right| \leq \frac{4s3s}{14s^2} = \frac{6}{7}.$$

We apply Boutroux-Cartan with $\gamma = \varepsilon S$ to find that outside a union U of at most n open discs of total radius at most $24\varepsilon s$,

$$\sum_{k=1}^{n} \log |z - z_k| \geq n \log 4\varepsilon s.$$

Thus, for $|z| \leq 2s$, $z \notin U$,

$$\begin{aligned}\log|h(z)| &= \log|g(z)| + \sum_{j=1}^{m} \log\left|\frac{S(z-a_j)}{S^2 - \overline{a_j}z}\right| + \sum_{k=1}^{n} \log S - \sum_{k=1}^{n} \log|z - z_k| \\ &\leq 6T(8s, h) + m \log\frac{6}{7} - n\log 4\varepsilon s + n \log 4s \\ &= 6T(8s, h) + m\log\frac{6}{7} - n\log\varepsilon\end{aligned}$$

where, by Lemma 1,

$$n = n(4s, h) \leq \frac{N(8s, h)}{\log 2} \leq \frac{T(8s, h)}{\log 2},$$

from which the result follows.

QED

The following lemma allows us to say that if a function has some zeros in a certain segment of the real line, then it has more zeros in a larger segment. Repeated application of this allows us to cover the entire range $[1, \infty)$.

Lemma 4

Given $d \in (0, 1)$, there exists $\vartheta = \vartheta(d) > 0$ with the following property. Let $R \geq 1$, $E \subseteq \mathbb{N}$ be such that $|E \cap [1, r]| \geq dr$ for all $r \geq R$, let $F(E) \subseteq \mathbb{Z}$ where F is meromorphic in $\mathbb{C}$ and has at least $dR/2$ distinct zeros in $E \cap [1, R]$, and $T(r, F) \leq \vartheta r$ for all $r \geq R$. Then F has at least dR distinct zeros in $E \cap [1, 2R]$.

Proof:

Let $\varepsilon = d/96$, and let m be the least integer such that $m \geq dR/2$. We apply Lemma 3 with $h = F$ and $s = R$ to give, for $|z| \leq 2R$ outside some union U of at most $n(4R, F)$ open discs of total radius at most $dR/4$,

$$\log|F(z)| \leq \left(6 - \frac{\log\varepsilon}{\log 2}\right) 8\vartheta R + \frac{dR}{2}\log\frac{6}{7}. \tag{3.3}$$

It is easy to check that with small enough ϑ, this gives $\log|F(z)| < 0$. Further, by our earlier remark on Lemma 3, U encloses at most

$$n(4R, F) + 48\varepsilon R \leq \frac{T(8R, F)}{\log 2} + 48\varepsilon R \leq \left(\frac{8\vartheta}{\log 2} + \frac{d}{2}\right) R \tag{3.4}$$

integers. Given that $|E \cap [1, 2R]| \geq 2dR$, it is clear that if ϑ is small enough then after removing any points of $E \cap [1, 2R] \cap U$ we are left with at least dR integers in $(E \cap [1, 2R]) \setminus U$, which, since $F(E) \subseteq \mathbb{Z}$ and $|F(z)| < 1$ at these points, must be zeros of F.

QED

We now proceed to several lemmas from [2], which form the main structure of the proof. We first create a sequence of polynomials, then look at an application of linear forms, and finally note that if a function is algebraic on a half plane and takes integer values, then it is a polynomial.

Lemma 5 [2]

Define polynomials $p_0, p_1, \ldots$ *by*

$$p_0(z) = 1, \quad p_1(z) = z, \quad p_h(z) = \frac{z(z-1)\ldots(z-h+1)}{h!} \ (h = 2, 3, \ldots).$$

Then for $R > 0$, $H \in \mathbb{N}$, $0 \leq h \leq H$ *and* $|z| \leq R$, *we have* $p_h(\mathbb{Z}) \subseteq \mathbb{Z}$ *and*

$$|p_h(z)| \leq e^H \left(\frac{R}{H} + 1 \right)^H .$$

Proof:

It is easy to see that $p_h(\mathbb{Z}) \subseteq \mathbb{Z}$. For the inequality, we write

$$|p_h(z)| \leq \frac{(R+H)^h}{h!} \leq \frac{H^h}{h!} \left(\frac{R}{H} + 1 \right)^H \leq e^H \left(\frac{R}{H} + 1 \right)^H .$$

QED

Lemma 6 [2]

Let $B \geq 1$ *and* $N \geq 2$ *be integers. Suppose that* $L_1, \ldots, L_m$ *are linear forms in the* n *variables* $x_1, \ldots, x_n$, *with real coefficients* $a_{j,k}$ *for* $j = 1, \ldots, m$ *and* $k = 1, \ldots, n$, *such that* $L_j = a_{j,1}x_1 + \ldots + a_{j,n}x_n$. *Suppose further that* $n > m$ *and*

$$\max_{j,k} |a_{j,k}| \leq B.$$

Then there exist integers $x_1, \ldots, x_n$, *not all zero, such that for* $j = 1, \ldots, m$ *and* $k = 1, \ldots, n$,

$$|L_j| \leq \frac{1}{N} \qquad \text{and} \qquad |x_k| \leq 2(2nBN)^{\frac{m}{n-m}}.$$

Lemma 7 [2]

Let the algebraic function f *be analytic on the half plane* $Re(z) \geq 0$, *and satisfy* $f(E) \subseteq \mathbb{Z}$ *for some set* $E \subseteq \mathbb{N}$ *of positive lower density. Then* f *is a polynomial.*

4 Proof of the theorem

Fix a large positive integer J, and given J let R be a large positive integer. How large J must be will be determined later.

Apply Lemma 3 with $h = f$, $m = 0$, $s = R/2$ and $\varepsilon = d/96$ to give that, for $|z| \leq R$ outside a union U of open discs of total radius at most $dR/8$,

$$\log |f(z)| \leq \left(6 - \frac{\log \frac{d}{96}}{\log 2}\right) 4\lambda R = \Lambda R. \tag{4.1}$$

By (3.4), replacing R with $R/2$,

$$|\mathbb{Z} \cap U| \leq \frac{T(4R, f)}{\log 2} + 24\varepsilon R \leq \left(\frac{4\lambda}{\log 2} + \frac{d}{4}\right) R < \frac{dR}{3} \tag{4.2}$$

for small enough λ. Since R is large we therefore have $m \geq dR/2$ distinct integers $\alpha_1, \ldots, \alpha_m \in E \cap [1, R]$, where $m/J \in \mathbb{N}$, for which $f(z) \in \mathbb{Z}$ and (4.1) is satisfied.

Now, set $n = 2m$, $H = n/J \in \mathbb{N}$, and form $n = HJ$ functions

$$g_k(z) = p_{\mu(k)}(z) f(z)^{\nu(k)}, \tag{4.3}$$

for $\mu = 0, 1, \ldots, H-1$, $\nu = 0, 1, \ldots, J-1$, where the p_μ are as in Lemma 5. Note that H is dependent on R, but that J is fixed. Let $a_{j,k} = g_k(\alpha_j) \in \mathbb{Z}$. We obtain the following estimate by Lemma 5 and (4.1):

$$\begin{aligned} |a_{j,k}| &= |g_k(\alpha_j)| = \left|p_{\mu(k)}(\alpha_j)\right| |f(\alpha_j)|^{\nu(k)} \\ &\leq e^H \left(\frac{R}{H} + 1\right)^H \left(e^{\Lambda R}\right)^{J-1} \\ &= A(R) \leq \lceil A(R) \rceil = B(R) \leq 2A(R), \end{aligned}$$

where $\lceil x \rceil$ is the smallest integer not less than x. We apply Lemma 6 with $N = 2$ and $n = 2m$ to give integers $A_1, \ldots, A_n$, not all zero, such that

$$\sum_{k=1}^{n} A_k g_k(\alpha_j) = 0$$

for $j = 1, \ldots, m$, and

$$|A_k| \leq 8nB, \text{where } B = B(R).$$

Now set

$$F(z) = \sum_{k=1}^{n} A_k g_k(z). \tag{4.4}$$

F is meromorphic, takes integer values on E and is 0 at the α_j for $j = 1, \ldots, m$. We now estimate $T(r, F)$ for each $r \geq R$. Note first that since the $p_\mu(z)$ are polynomials, all poles of F must come from poles of f, and so

$$N(r, F) \leq (J-1)N(r, f).$$

Also, for non-negative $x_1, \ldots, x_n$,

$$\log^+ \left(\sum_{k=1}^{n} x_k \right) \leq \log n + \max_{1 \leq k \leq n} \log^+ |x_k|.$$

For $r \geq R$, we have by Lemma 5 that

$$\begin{aligned} \log |F(z)| &\leq \log n + \max_{1 \leq k \leq n, |z|=r} \left(\log^+ |A_k g_k(z)| \right) \\ &\leq \log n + \log 8nB + H \left(1 + \log \left(\frac{r}{H} + 1 \right) \right) + (J-1) \log^+ |f(z)|. \end{aligned}$$

Thus, by integrating we obtain

$$m(r, F) \leq \log n + \log 8nB + H \left(1 + \log \left(\frac{r}{H} + 1 \right) \right) + (J-1)m(r, f),$$

and so

$$\begin{aligned} T(r, F) &\leq \log n + \log 8nB + H \left(1 + \log \left(\frac{r}{H} + 1 \right) \right) + (J-1)T(r, f) \\ &\leq \log n + \log 16nA + H \left(1 + \log \left(\frac{r}{H} + 1 \right) \right) + (J-1)T(r, f). \end{aligned}$$

Now, for $r \geq R$, since $\Lambda > \lambda$ by (4.1) and $n = 2m \leq 2r$ and R is large, we have

$$\begin{aligned} T(r, F) &\leq 4 \log 2 + 2 \log n + \log \left(e^H \left(\frac{R}{H} + 1 \right)^H e^{(J-1)\Lambda R} \right) + \\ &\qquad + H \left(1 + \log \left(\frac{r}{H} + 1 \right) \right) + (J-1)T(r, f) \\ &\leq 4 \log 2 + 2 \log 2r + H \left(1 + \log \left(\frac{R}{H} + 1 \right) \right) + (J-1)\Lambda R + \\ &\qquad + H \left(1 + \log \left(\frac{r}{H} + 1 \right) \right) + (J-1)\lambda r \\ &\leq 2H \left(1 + \log \left(\frac{r}{H} + 1 \right) \right) + 2(J-1)\Lambda r. \end{aligned}$$

It is plain to see by differentiating that $x^{-1}(1 + \log(x+1))$ is decreasing for $x > 0$. So, for $n = 2m \leq 2R \leq 2r$, this gives

$$\frac{r}{H} \geq \frac{R}{H} = \frac{RJ}{n} = \frac{RJ}{2m} \geq \frac{J}{2}$$

and

$$\begin{aligned} 2H\left(1+\log\left(\frac{r}{H}+1\right)\right) &= 2r\frac{H}{r}\left(1+\log\left(\frac{r}{H}+1\right)\right) \\ &\leq 2r\frac{2}{J}\left(1+\log\left(\frac{J}{2}+1\right)\right) \\ &= \frac{4r}{J}\left(1+\log\left(\frac{J}{2}+1\right)\right). \end{aligned}$$

Thus,

$$T(r,F) \leq \frac{4r}{J}\left(1+\log\left(\frac{J}{2}+1\right)\right) + 2(J-1)\Lambda r, \tag{4.5}$$

and so we can say that for large enough R,

$$T(r,F) < \vartheta r \tag{4.6}$$

for $r \geq R$, where $\vartheta > 0$ can be arbitrarily small provided that λ is small enough and J large enough. We also have $F(\alpha_j) = 0$ for $j = 1, \ldots, m$ where $m \geq dR/2$. We apply Lemma 4 to give at least dR zeros of F in $E \cap [1, 2R]$. We apply this repeatedly to give an infinite sequence of zeros of F on the real line. Assume that $F(z) \not\equiv 0$. We have that $n(2^t R, 1/F) \geq 2^{t-1} dR$, and so $n(r, 1/F) \geq dr/4$ for all $r \geq R$. By application of Lemma 1 we find $N(er, 1/F) \geq dr/4$, thus $T(r, 1/F) \geq dr/4e$, and so by the First Fundamental Theorem,

$$T(r,F) \geq dr/4e - O(1). \tag{4.7}$$

However, if ϑ is small enough, this is incompatible with (4.6). Hence, $F(z) \equiv 0$.

Now, recall from (4.3) and (4.4) that

$$F(z) = \sum_{\nu=0}^{J-1}\left(\sum_{\mu=0}^{H-1} A_{\mu,\nu} p_\mu(z)\right) f(z)^\nu,$$

where at least one $A_{\mu,\nu}$ is non-zero, and where $p_\mu(z)$ has degree μ. Thus, these polynomials cannot cancel each other out, hence f is algebraic, and so must have only finitely many poles. Therefore there is some $x \in \mathbb{N}$ such that there are no poles in the half plane $Re(z) \geq x$, so f is analytic in this region. We apply Lemma 7 to $f(z-x)$, giving that $f(z-x)$ is a polynomial here. From this, we conclude that $f(z)$ is polynomial in the half-plane $Re(z) \geq x$, and thus by the identity theorem $f(z)$ must be a polynomial on the whole plane.

QED

5 Appendix - How small is $\lambda(d)$?

An obvious question to ask about this theorem is "how small must λ be?" We will now calculate this.

We begin by calculating ϑ in Lemma 4. We use (3.3), substituting in $d/96$ for ε, and noting that since we want $|F(z)| < 1$ in order to force $F(\alpha) = 0$ for $\alpha \in E \setminus U$, we require $\log |F(z)| < 0$. Hence,

$$\vartheta < \frac{d \log \frac{7}{6}}{16\left(6 - \frac{\log(d/96)}{\log 2}\right)} = \gamma(d). \tag{5.1}$$

We also require that U encloses at most dR integers, and so by (3.4) we need

$$\left(\frac{8\vartheta}{\log 2} + \frac{d}{2}\right) R \leq dR,$$

which simplifies to

$$\vartheta \leq \frac{d \log 2}{16}. \tag{5.2}$$

We further require by (4.6) and (4.7) that

$$\vartheta < \frac{d}{4e}. \tag{5.3}$$

However,

$$\frac{d \log \frac{7}{6}}{16\left(6 - \frac{\log(d/96)}{\log 2}\right)} < \frac{d \log \frac{7}{6}}{96} < \frac{d}{48} < \frac{d \log 2}{16} < \frac{d}{4e},$$

hence both (5.2) and (5.3) are much looser bounds than (5.1) and so may be ignored.

We now move on to Λ. The proof of the theorem by (4.2) requires

$$\lambda < \frac{d \log 2}{48}. \tag{5.4}$$

It also requires by (4.1) and (4.5) that

$$\vartheta = \frac{4}{J}\left(1 + \log\left(\frac{J}{2} + 1\right)\right) + 2(J-1)\left(6 - \frac{\log \frac{d}{96}}{\log 2}\right) 4\lambda.$$

Suppose we choose J so large that

$$\frac{4}{J}\left(1 + \log\left(\frac{J}{2} + 1\right)\right) < \frac{\gamma(d)}{2},$$

and, given this J, choose λ such that

$$2(J-1)\left(6-\frac{\log\frac{d}{96}}{\log 2}\right)4\lambda<\frac{\gamma(d)}{2}.$$

Then the pair (J,λ) will satisfy (5.1). Further, we have

$$\lambda<\frac{\gamma(d)}{96}<\frac{d\log\frac{7}{6}}{96^2}<\frac{d\log 2}{48},$$

and so (5.4) holds. Solving these inequalities using Mathematica for J in terms of d produces the following new inequality:

$$J>\frac{128\log\frac{d}{6144}}{d\log\frac{7}{6}\log 2}W\left(\frac{d\log\frac{7}{6}\log 2\exp\left(\frac{d\log\frac{7}{6}\log 2}{64\log\frac{d}{6144}}-1\right)}{64\log\frac{d}{6144}}\right)-2$$

where W is the Lambert W-function. Again using Mathematica, solving for specific values of d gives the following results for J and λ:

$d=1$	$J\gtrapprox 130,000$	$\lambda\lessapprox 2.9\times 10^{-11}$
$d=0.5$	$J\gtrapprox 290,000$	$\lambda\lessapprox 5.6\times 10^{-12}$
$d=0.1$	$J\gtrapprox 2,000,000$	$\lambda\lessapprox 1.2\times 10^{-13}$
$d=0.01$	$J\gtrapprox 28,000,000$	$\lambda\lessapprox 5.8\times 10^{-16}$

Note that $d=1$ is essentially meaningless here, as we require our set E to have lower density greater than d, but it provides a useful upper bound.

By comparison, using a similar process on the Fletcher-Langley result (Proposition 3) yields a maximal value of λ of roughly 3.6×10^{-4} for d close to 1.

We include by asking a question about restricting the integer values taken. For $n\in\{1,2,4\}$, is 2^{nz} is the slowest-growing transcendental meromorphic function taking only n^{th} powers of integers on the natural numbers? Pólya's result proves this for $n=1$, but beyond this the way forward is unclear. The restriction to only three integer values of n is due to the sine function: for odd $n\geq 3$, $\sin(\pi z/2)$ is smaller than 2^{3z} and has the required properties, and for even $n\geq 6$, $\sin(\pi z)$ is sufficient.

References

[1] R. C. Buck, *Integral valued entire functions*, (Duke Mathematical Journal, 15, no.4, 1948, pp879-891).

[2] A. N. Fletcher and J. K. Langley, *Integer points of analytic functions in a half plane*, (Proceedings of the Edinburgh Mathematical Society (2), 52, no.3, 2009, pp619-630).

[3] W. K. Hayman, *Subharmonic Functions Volume 2*, (London, Academic Press, 1989).

[4] I. Laine, *Complex Analysis 2*, (*http://www.joensuu.fi/matematiikka/kurssit/complex/*, retrieved August 2011).

[5] J. K. Langley, *Integer points of entire functions*, (Bulletin of the London Mathematical Society, 38, 2006, pp239-249).

[6] C. Pisot, *Über ganzwertige ganze Funktionen* (Jahresbericht der Deutsche Mathematiker-Vereinigung, 52, 1942, pp95-102).

[7] G. Pólya, *Über ganze ganzwertige Funktionen*, (Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, c1920, pp1-10).

[8] R. M. Robinson, *Integer-valued entire functions*, (Transactions of the American Mathematical Society, 153, 1971, pp451-468).

[9] A. Selberg, *Über ganzwertige ganze transzendente Funktionen*, (Archiv for Mathematik og Naturvidenskab, 44, 1941, pp45-52).

[10] M. Waldschmidt, *Integer valued entire functions on Cartesian products*, (Number Theory in Progress, 1, 1999, pp553-576).

[11] K. Yoshino, *Polya's theorem for nonentire functions*, (Rendiconti del Circolo Matematico di Palermo (2), no.3, 1984, pp385-395).

Address:
Matthew Buck
School of Mathematical Sciences
University Park
The University of Nottingham
Nottingham
NG7 2RD

E-mail:
pmxmmb@nottingham.ac.uk