A theorem on homogeneous differential polynomials

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Abstract

We substantially strengthen an unpublished result of Whitehead from his PhD thesis [8] using a refinement of his techniques.

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1 Introduction and definitions

Let M_j be a differential monomial in a function f, meromorphic in the plane, given by

$$M_j[f] = f^{\mu_{0,j}} \left(f'\right)^{\mu_{1,j}} \dots \left(f^{(q)}\right)^{\mu_{q,j}}, \tag{1.1}$$

of degree

$$\gamma_j = \mu_{0,j} + \dots + \mu_{q,j},$$

and weight

$$\Gamma_j = \mu_{0,j} + 2\mu_{1,j} + \dots + (q+1)\mu_{q,j}.$$

Where we write meromorphic, it is implied to mean meromorphic in the plane.

We consider sums of monomials M_j of equal degree n, forming homogeneous differential polynomials, and consider what we may deduce about the function f from knowing properties of this polynomial. There are numerous results on homogeneous

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differential polynomials, including [1], [2], [6] and [7]. We use the standard notation of [4] throughout.

2 The main result

Theorem

Let f be nonconstant and meromorphic in the plane. Let $u = f/f'$, $m \in \mathbb{N}$, and let F, a homogeneous differential polynomial in f and its derivatives of weight Γ_F, be defined by

$$F = f^n + \sum_{j=1}^{m} a_j M_j[f] \not\equiv 0, \tag{2.1}$$

where the $M_j[f]$ are as defined in (1.1) with degree n, and the a_j are small functions with respect to u such that $T(r, a_j) = S(r, u)$. Suppose that

$$\mu_{0,j} \notin \{n-1, n\} \; \forall j, \tag{2.2}$$

and that

$$\Gamma_F \geq 2n. \tag{2.3}$$

Then at least one of the following must hold:

1. *We have that*

$$f = Re^P \tag{2.4}$$

for some rational function R and polynomial P in z;

2. *We have that*

$$F \equiv f^n; \tag{2.5}$$

3. *The following inequality holds:*

$$T(r, u) \leq (\Gamma_F - 3)\overline{N}_1\left(r, \frac{1}{f'}\right) + \overline{N}_2\left(r, \frac{1}{F}\right) + S(r, u), \tag{2.6}$$

where $\overline{N}_1(r, 1/f')$ counts zeros of f' which are not also zeros of f, without regard to multiplicity, and $\overline{N}_2(r, 1/F)$ counts zeros of F which are neither poles nor zeros of f, again without regard to multiplicity.

Remark:

Conditions (2.2) and (2.3) force $n \geq 2$ and thus $\Gamma_F \geq 4$.

This result is an improvement of Whitehead's Theorem 5.16 [8]. Whitehead's result required that there be a term of unique maximal weight (see the remark in the proof of the main theorem for details), replaced condition (2.3) with the stricter requirement that there be some k such that $\mu_{0,k} = 0$, and had the inequality

$$T(r, u) \leq \overline{N}\left(r, \frac{1}{f}\right) + (\Gamma_F - 3)\overline{N}\left(r, \frac{1}{f'}\right) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, u)$$

in place of (2.6).

Condition (2.2) is necessary for us to apply this theorem (again, see the remark in the proof of the main theorem for details), since we can construct examples without this assumption such that none of the conclusions of the above theorem hold. For instance, let $F = f^2 + 2ff'' - 3(f')^2$, with

$$f = \frac{1}{1 - e^z}, \qquad f' = \frac{e^z}{(1 - e^z)^2}, \qquad f'' = \frac{e^z(1 + e^z)}{(1 - e^z)^3}.$$

Then $F = f^4 \neq f^2$, and so conclusion 2 does not hold. Further, neither F nor f' have any zeros, and hence conclusion 3 cannot hold. Since conclusion 1 clearly does not hold, our example is such that none of the conclusions hold. Another example using the same f is $F = f^3 + 3f^2f'' + 7(f')^3 - 3ff'f''$.

Corollary

Let the hypotheses of the main theorem hold, with the additional condition that $m = 1$ in (2.1) and $F = f^n + aM[f]$ where $a \not\equiv 0$. Then at least one of (2.4) and (2.6) holds.

Remark:

Whitehead had a similar corollary for his theorem.

3 Lemmata

3.1 An inequality related to a result of Zhang and Li

The first part of working towards a proof of the main theorem is to improve a result of Zhang and Li of Tumura-Clunie type. First though, we need a lemma.

Lemma 1 - Clunie's Lemma [3]

Suppose that $h^n P[h] = Q[h]$, where h is meromorphic and $P[h]$ and $Q[h]$ are polynomials in h and its derivatives with meromorphic functions satisfying $m(r, c) = S(r, h)$ as coefficients, $Q[h]$ being of degree n at most. Then,

$$m(r, P[h]) = S(r, h). \tag{3.1}$$

Theorem 2

Let the function h be non-constant and meromorphic in the plane. Assume that

$$\psi = h^n + P[h] \not\equiv 0, \tag{3.2}$$

where $P[h]$ is a differential polynomial in h with coefficients c_j which are small functions with respect to h, i.e. $T(r, c_j) = S(r, h)$. Suppose that $P[h]$ has degree at most $n-2$; then at least one of the following is true:

1. *We have*
$$T(r, h) < (\Gamma_P - n + 3)\,\overline{N}(r, h) + \overline{N}_0\left(r, \frac{1}{\psi}\right) + S(r, h) \tag{3.3}$$
as $r \to \infty$, where $\overline{N}_0(r, 1/\psi)$ counts zeros of ψ which are not zeros of h, without regard to multiplicity, and Γ_P is the weight of P;

2. *We have*

$$\psi \equiv h^n. \tag{3.4}$$

Remark:

This is an improvement of Whitehead's Theorem 5.8 [8], based on the methods of Zhang and Li [9], where it makes up the final two pages of their proof, but is not itself presented as a result. Whitehead's version had $\overline{N}(r, 1/\psi)$ instead of $\overline{N}_0(r, 1/\psi)$. This improvement will be very important later.

We also note here that this result still holds even if $\Gamma_P - n + 3$ is negative.

Proof:

Differentiating (3.2), we get $\psi' = nh^{n-1}h' + P'$, and so

$$\frac{\psi'}{\psi}(h^n + P) = nh^{n-1}h' + P',$$

$$\begin{aligned} \frac{\psi'}{\psi}P - P' &= nh^{n-1}h' - \frac{\psi'}{\psi}h^n \\ &= h^{n-1}H, \end{aligned} \tag{3.5}$$

where

$$H = nh' - \frac{\psi'}{\psi}h. \tag{3.6}$$

Case I:

Suppose that $H \not\equiv 0$. Then since $m(r, \psi'/\psi) = S(r, \psi) = S(r, h)$ by the Lemma of the Logarithmic Derivative [4], we may use Clunie's Lemma, giving that

$$m(r, H) = S(r, h). \tag{3.7}$$

Since P has degree at most $n - 2$ and P' has at most the same degree, we can write (3.5) as

$$h^{n-2}(hH) = \frac{\psi'}{\psi}P - P',$$

and use Clunie's Lemma to give $m(r, hH) = S(r, h)$. Using this, the First Fundamental Theorem and (3.7), we get

$$\begin{aligned} m(r,h) &= m\left(r, \frac{hH}{H}\right) \\ &\leq m\left(r, \frac{1}{H}\right) + m(r, hH) \\ &= T(r,H) - N\left(r, \frac{1}{H}\right) + S(r,h) \\ &= N(r,H) - N\left(r, \frac{1}{H}\right) + S(r,h), \end{aligned} \tag{3.8}$$

and hence

$$T(r,h) \leq N(r,h) + N(r,H) - N\left(r, \frac{1}{H}\right) + S(r,h). \tag{3.9}$$

Let $z_0 \in \mathbb{C}$, and suppose that at z_0, h has a pole of order $q \geq 0$, and let its contribution to $n(r,H) - n(r,1/H)$ be $-t$, so that $t > 0$ if z_0 is a zero of H and $t < 0$ if z_0 is a pole of H.

If $q \geq 1$, then a monomial $M = ch^{i_0}\left(h'\right)^{i_1} \ldots \left(h^{(k)}\right)^{i_k}$ has a pole of order at most $qi_0 + (q+1)i_1 + \ldots + (q+k)i_k + s$ where the s is the contribution from the coefficient c. This is

$$\begin{aligned} \sum_{j=0}^{k}(j+1)i_j + (q-1)\sum_{j=0}^{k} i_j + s &= \Gamma_M + (q-1)\gamma_M + s \\ &\leq \Gamma_P + (q-1)(n-2) + s, \end{aligned} \tag{3.10}$$

since Γ_P is the maximum of the Γ_M over all terms in P. We now rewrite (3.5) in the form

$$h^{n-1} = \left(\frac{\psi'}{\psi}P - P'\right)\frac{1}{H}.$$

Then we have

$$(n-1)q \leq \Gamma_P + (q-1)(n-2) + s + 1 + t,$$

where the 1 comes from ψ'/ψ having at most a simple pole, and t is as defined above. Thus,

$$q - t \leq \Gamma_P - n + 3 + s,$$

and hence z_0 contributes at most $\Gamma_P - n + 3 + s$ to

$$n_1(r) = n(r, h) + n(r, H) - n(r, 1/H) \tag{3.11}$$

and at least $\Gamma_P - n + 3$ to

$$n_2(r) = (\Gamma_P - n + 3)\overline{n}(r, h) + \overline{n}_0\left(r, \frac{1}{\psi}\right), \tag{3.12}$$

where $\overline{n}_0(r, 1/\psi)$ counts the distinct points at which $\psi = 0$ but $h \neq 0$.

Now suppose that $q = 0$ but $t \neq 0$. If $t > 0$, then the contribution to $n_1(r)$ is negative and the contribution to $n_2(r)$ is non-negative. If $t < 0$, then z_0 must be a simple pole of H arising from the term ψ'/ψ in (3.6). Such a simple pole of H can be caused by a zero of ψ which is not a zero of h, which then gives $t = -1$ and contributes 1 to each of $n_1(r)$ and $n_2(r)$. The only other possibility is a pole of ψ caused by a pole of the coefficients, which contributes 1 to $n_1(r)$ and 0 to $n_2(r)$, and the number of such poles is $S(r, h)$. Thus (3.9) becomes (3.3).

Case II:

Assume now that $H \equiv 0$. Then by (3.5), we have $P = \lambda\psi$ for some $\lambda \in \mathbb{C}$. If $\lambda = 0$ then $P \equiv 0$ and so (3.4) follows by (3.2). Now suppose that $\lambda \neq 0$, and let $\Lambda = \lambda^{-1}$. Then we have $h^n + P = \psi = \Lambda P$, and so

$$h^{n-1}h = (\Lambda - 1)P.$$

We apply Clunie's Lemma, giving

$$m(r, h) = S(r, h). \tag{3.13}$$

We further note that

$$h^2 = (\Lambda - 1)Ph^{2-n},$$

and look at poles of this. As before, if h has a pole of order $q \geq 1$, then by (3.10), P has a pole of order at most $\Gamma_P + (q-1)(n-2) + s$, and so Ph^{2-n} will have a pole of order at most

$$\Gamma_P + (q-1)(n-2) + s - q(n-2) = \Gamma_P - (n-2) + s.$$

Using (3.13),

$$\begin{aligned} T(r,h) = N(r,h) + S(r,h) &= \frac{1}{2}N(r,h^2) + S(r,h) \\ &\leq \frac{1}{2}\left(\Gamma_P - (n-2)\right)\overline{N}(r,h) + S(r,h). \end{aligned} \tag{3.14}$$

With $\Gamma_P > n-2$, it is easy to see that this implies (3.3). If however, $\Gamma_P \leq n-2$, then by (3.13) and (3.14) we have $T(r,h) = S(r,h)$, a contradiction.

QED

3.2 Several lemmas by Whitehead

We now present several lemmas from [8]. We include the proofs for completeness, as Whitehead's thesis is unpublished. We begin with a result comparing the weight of a monomial with the weight of its derivative.

Lemma 2 [8]

Let $M[u]$ be a monomial as defined in (1.1). If $M[u]$ has weight Γ_M then $M'[u]$ has weight $\Gamma_M + 1$.

This result is proved my induction on q, the highest derivative of u occurring in $M[u]$. We now show that we may write higher derivatives of f in terms of f and u.

Lemma 3 [8]

Let $p \in \mathbb{N}$ and $u = \frac{f}{f'}$. Then,

$$f^{(p)} = f\frac{S_p[u]}{u^p}, \tag{3.15}$$

where

$$S_1[u] = 1$$
$$S_2[u] = 1-u'$$

and

$$S_p[u] = (1-u')(1-2u')\dots(1-(p-1)u') + uT_{p-2}[u] \tag{3.16}$$

for $p \geq 3$*, with* $T_{p-2}[u]$ *a differential polynomial in* u *with constant coefficients and degree at most* $p-2$*, such that* $T_0[u] \equiv 0$*. Further, each* $S_p[u]$ *has degree* $p-1$ *and weight* $2(p-1)$*.*

Proof:

We begin by noting that

$$f' = \frac{f}{u} = \frac{f}{u}S_1[u] \qquad \text{and} \qquad f'' = \frac{f'}{u} - \frac{fu'}{u^2} = \frac{f}{u^2}S_2[u],$$

and thus the lemma holds for $p \in \{1,2\}$. Assume it holds for some $p = k \geq 2$, then we have

$$\frac{f^{(k)}}{f} = \frac{S_k[u]}{u^k},$$

and hence

$$\begin{aligned} \frac{f^{(k+1)}}{f} &= \left(\frac{f^{(k)}}{f}\right)' + \frac{f^{(k)}}{f}\frac{f'}{f} \\ &= \left(\frac{S_k[u]}{u^k}\right)' + \frac{S_k[u]}{u^k}\frac{1}{u} \\ &= \frac{S_{k+1}[u]}{u^{k+1}}, \end{aligned}$$

where

$$S_{k+1}[u] = uS_k'[u] + (1-ku')S_k[u]. \tag{3.17}$$

We now prove (3.16). Substituting into (3.17), we have

$$S_{k+1}[u] = (1-u')\dots(1-(k-1)u')(1-ku') + (1-ku')uT_{k-2}[u] + uS_k'[u],$$

and we set

$$T_{k-1}[u] = (1 - ku')T_{k-2}[u] + S'_k[u]$$

which has degree at most

$$\max\{1 + (k-2),\, k-1\} = k - 1 = (k+1) - 2.$$

Also, (3.17) and Lemma 2 show that $S_{k+1}[u]$ has weight at most

$$\max\{\Gamma_{S_k} + 2,\, \Gamma_{S_k},\, \Gamma_{S_k} + 2\} = 2(k-1) + 2 = 2((k+1) - 1),$$

and the presence of the term $(1 - u') \dots (1 - ku')$ in $S_{k+1}[u]$ shows that the degree of $S_{k+1}[u]$ is $(k+1) - 1$ and the weight is $2((k+1) - 1)$.

QED

We now show that we may write a differential monomial in f in terms of f and a differential polynomial in u.

Lemma 4 [8]

Let the hypotheses of the main theorem hold, let $L = \Gamma_F - n$ and $u = \frac{f}{f'}$. Then

$$u^L \frac{M_j[f]}{f^n} = V_j[u] \tag{3.18}$$

where $V_j[u]$ is a differential polynomial in u of degree at most $L - 2$ and weight at most $2\Gamma_F - n - 6$.

Proof:

By the hypotheses, $\Gamma_F \geq 2n$, and so $L \geq n$. We apply Lemma 3 to (1.1),

$$\begin{aligned}
u^L \frac{M_j[f]}{f^n} &= u^L \prod_{p=1}^{q} \left(\frac{f^{(p)}}{f} \right)^{\mu_{p,j}} \\
&= u^L \prod_{p=1}^{q} \left(\frac{S_p[u]}{u^p} \right)^{\mu_{p,j}} \\
&= u^{\delta_j} \prod_{p=1}^{q} S_p[u]^{\mu_{p,j}} \\
&= V_j[u],
\end{aligned}$$

where

$$\begin{aligned}
\delta_j &= L - \sum_{p=0}^{q} p\mu_{p,j} \\
&= \Gamma_F - n - \sum_{p=0}^{q} p\mu_{p,j} \\
&= \Gamma_F - \sum_{p=0}^{q} (p+1)\mu_{p,j} \\
&= \Gamma_F - \Gamma_j \\
&\geq 0.
\end{aligned}$$

Since $\mu_{0,j} \notin \{n-1, n\}$, we have $\mu_{1,j} + \ldots + \mu_{q,j} \geq 2$. Thus using Lemma 3

$$\begin{aligned}
\gamma_{V_j} &\leq \delta_j + \sum_{p=1}^{q} (p-1)\mu_{p,j} \\
&= L - \sum_{p=1}^{q} (p\mu_{p,j} - (p-1)\mu_{p,j}) \\
&= L - \sum_{p=1}^{q} \mu_{p,j} \\
&\leq L - 2. \qquad (3.19)
\end{aligned}$$

Further, again by Lemma 3,

$$\begin{aligned}
\Gamma_{V_j} &\leq \delta_j + 2\sum_{p=1}^{q}(p-1)\mu_{p,j} \\
&= L - \sum_{p=1}^{q} p\mu_{p,j} + 2\sum_{p=1}^{q}(p-1)\mu_{p,j} \\
&= L + \sum_{p=1}^{q}(p+1)\mu_{p,j} - 3\sum_{p=1}^{q}\mu_{p,j} \\
&\leq L + \Gamma_j - 6 \\
&= \Gamma_F - n + \Gamma_j - 6 \\
&\leq 2\Gamma_F - n - 6. \qquad (3.20)
\end{aligned}$$

3.3 Some final lemmas

We conclude this section with two improvements to Whitehead's lemmas, and the statement of a standard result.

Lemma 5

We have

$$\overline{N}(r,u) \leq \overline{N}_1\left(r, \frac{1}{f'}\right), \qquad (3.21)$$

where the right hand term counts zeros of f' which are not also zeros of f, without regard to multiplicity.

Proof:

Since $u = f/f'$, all poles of u must come from poles of f or zeros of f'. But a pole of f or a zero of f' which is also a zero of f would result in a zero of u. Therefore all poles of u must come from zeros of f' which are not also zeros of f.

QED

The next lemma is a refinement of one of Whitehead's lemmas.

Lemma 6

Let F be as defined in (2.1), and $L = \Gamma_F - n$ as before. Further, let

$$\psi = \frac{u^L F}{f^n}. \tag{3.22}$$

Then we have

$$\overline{N}_0\left(r, \frac{1}{\psi}\right) \leq \overline{N}_2\left(r, \frac{1}{F}\right), \tag{3.23}$$

where $\overline{N}_0(r, 1/\psi)$ counts the zeros of ψ which are not zeros of u without regard to multiplicity, and $\overline{N}_2(r, 1/F)$ counts zeros of F which are neither poles nor zeros of f, again without regard to multiplicity.

Proof:

ψ could have a zero if if f has a pole, if F has a zero or if u has a zero. However since $u = f/f'$, any pole or zero of f is also a zero of u and thus is not counted by $\overline{N}_0(r, 1/\psi)$.

QED

Our final lemma is a well-known result, which we state without proof.

Lemma 7

If $u = f/f'$ is a rational function, then $f = Re^P$, where R is a rational function and P is a polynomial.

4 Proof of the main theorem

If u is rational, then by Lemma 7 we obtain the first conclusion (2.4). Suppose now that u is transcendental. Then by (2.1) and Lemma 4,

$$\psi = \frac{u^L F}{f^n} = u^L + \sum_{j=1}^{m} a_j u^L \frac{M_j[f]}{f^n} = u^L + \sum_{j=1}^{m} a_j V_j[u]. \tag{4.1}$$

Since $V_j[u]$ has degree at most $L-2$, we may apply Theorem 2; and so either $\psi \equiv u^L$, and thus $F \equiv f^n$; or

$$T(r,u) < (\max\{\Gamma_{V_j}\} - L + 3)\overline{N}(r,u) + \overline{N}_0\left(r, \frac{1}{\psi}\right) + S(r,u).$$

Thus, by Lemma 4,

$$\begin{aligned} T(r,u) &< (2\Gamma_F - n - 6 - L + 3)\overline{N}(r,u) + \overline{N}_0\left(r, \frac{1}{\psi}\right) + S(r,u) \\ &= (\Gamma_F - 3)\overline{N}(r,u) + \overline{N}_0\left(r, \frac{1}{\psi}\right) + S(r,u), \end{aligned}$$

from which (2.6) follows by Lemmas 5 and 6.

QED

Remark:

Whitehead's requirement that there be a term of unique maximal weight came from his version of Lemma 6, which did not ignore zeros of u. He noted that having a pole of f with two monomials of maximal weight could allow one to cancel the other out. However, since we ignore zeros of u, and any pole of f is a zero of u, we may safely ignore poles of f, and so this requirement can be disregarded.

The requirement (2.2) stems from the hypotheses of Theorem 2. If (2.2) does not hold, then we could have that

$$\mu_{1,j} + \ldots + \mu_{q,j} = 1,$$

which in Lemma 4 would give that $V_j[u]$ could have degree $L-1$, and so we would not be able to apply Theorem 2 in the above proof.

4.1 Proof of the corollary:

Using the main theorem, at least one of (2.4), (2.5) or (2.6) holds. Suppose that (2.5) holds, then

$$a \prod_{p=1}^{q} \left(f^{(p)}\right)^{\mu_p} \equiv 0,$$

and so $f^{(p)} \equiv 0$ for some $1 \leq p \leq q$, since $a \not\equiv 0$. Thus f is a polynomial and so satisfies (2.4).

QED

Remark: Whitehead proved a version of this for his theorem, the method is identical.

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