2nd year Research Report

School of Mathematical Sciences
University of Nottingham

Matthew Buck

Supervisor: Prof. J. K. Langley

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Contents

1 Introduction to Nevanlinna Theory

Nevanlinna Theory is primarily concerned with the study of meromorphic functions - functions which are complex differentiable at all points except an isolated set of poles. Such functions are often written in the form $f(z) = \dfrac{A(z)}{B(z)}$ for entire functions A and B such that the poles of f are the zeroes of B. Such functions cannot be analysed using standard tools such as the maximum modulus function due to their poles.

We first define the *multiplicity* of a zero z_0 of f as the least n such that the coefficient of $(z - z_0)^n$ in the Taylor expansion of f about z_0 is non-zero. Since if f has a zero, $1/f$ has a pole, we define the multiplicity of a pole z_1 of f as the multiplicity of the zero of $1/f$ at that point.

Using this definition, we may define the unintegrated counting function $n(r, f)$ as the number of poles of the meromorphic function f in $\overline{B}(0, r)$, counting multiplicity. We then define the (Integrated) Counting Function, $N(r, f)$, as

$$N(r, f) = \int_0^r \left[n(t, f) - n(0, f)\right] \frac{dt}{t} + n(0, f) \log r.$$

This is the first of the Nevanlinna functionals. We also define the Proximity Function, $m(r, f)$, as

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta,$$

and from these two functions we define the Nevanlinna Characteristic, $T(r, f)$, by

$$T(r, f) = m(r, f) + N(r, f).$$

Finally, we write $S(r, f)$ for any term which is $o\left(T(r, f)\right)$.

2 Literature review and research overview

After finishing my work of last year, generalising Langley's work on pairs of non-homogeneous linear differential polynomials [18] and submitting my results [4], I started reading Zalcman's paper *Normal families: new perspectives* [30], Bergweiler's paper *Bloch's Principle* [1], his 2003 collaboration with Langley *Nonvanishing derivatives and normal families* [2], and Clifford's paper *Two New Criteria for Normal Families* [5], with a view to possibly generalising Bergweiler and Langley's result in some manner. However, I was not drawn to the concept of normal families, and so dropped that line of research.

I instead started looking at chapter 5 of Andrew Whitehead's PhD thesis from 2002 [29], entitled *Differential Equations and Differential Polynomials in the Complex Plane*. In this, Whitehead used Clunie's Lemma [7] to prove the following result:

Proposition [29]

Let $q \geq 1$, $n \geq 2$ *and that*

$$F = f^n + \sum_{j=1}^{m} \alpha_j M_j[f]$$

where f *and the* α_j *are meromorphic in the plane subject to the following conditions:*

- $\alpha_j \not\equiv 0$ and $T(r, \alpha_j) = S(r, f/f')$.
- *Each* M_j *is a monomial of the form*

  $$M_j[f] = f^{\beta_{0,j}} (f')^{\beta_{1,j}} \dots \left(f^{(q)}\right)^{\beta_{q,j}},$$

  with degree $\gamma_j = \sum_{k=0}^{q} \beta_{k,j} = n$, *and weight* $\Gamma_j = \sum_{k=0}^{q} (k+1)\beta_{k,j}$.
- $\beta_{0,j} = 0$ *for at least one* j, *and* $\beta_{0,j} \notin \{n-1, n\} \; \forall j$.
- *There exists a* t *such that* $\Gamma_j < \Gamma_t \; \forall j \neq t$. *The weight of* F, Γ_F, *is equal to the weight of the largest term, and therefore* $\Gamma_F = \Gamma_t$.

Then at least one of the following must hold:

- $f = Re^P$ *for some polynomial* P *and rational function* R *in* z*.*
- $F \equiv f^n$.
- $T\left(r, \dfrac{f}{f'}\right) \leq (\Gamma_F - 3)\overline{N}\left(r, \dfrac{1}{f'}\right) + \overline{N}\left(r, \dfrac{1}{f}\right) + \overline{N}\left(r, \dfrac{1}{F}\right) + S\left(r, \dfrac{f}{f'}\right).$

I was able to refine Whitehead's result:

Theorem A

Let the definitions of the above proposition hold, and let $\overline{N}_1(r, 1/f')$ *count zeros of* f' *which are not also zeros of* f*, without regard for multiplicity. Then at least one of the following must hold:*

- $f = Re^P$ *for some polynomial* P *and rational function* R *in* z*.*
- $F \equiv f^n$.
- $T\left(r, \dfrac{f}{f'}\right) \leq (\Gamma_F - 3)\overline{N}_1\left(r, \dfrac{1}{f'}\right) + \overline{N}\left(r, \dfrac{1}{F}\right) + S\left(r, \dfrac{f}{f'}\right).$

I was also able to improve on a corollary of Whitehead's, and show that if $m = 1$ then $F \not\equiv f^n$.

One of Whitehead's lemmas stated that if the quotient of a function and its derivative f/f' is a rational function, then the function may be written in the form $f = Re^P$ for some polynomial P and rational function R. I was able to improve this to show that if the quotient of $f/f^{(k)}$ is a rational function and either f or $f^{(k)}$ have only finitely many zeros then $f = Re^P$. I then applied this to non-linear homogeneous differential polynomials ψ in f to find sufficient conditions such that f may be written in the form Re^P. I also found conditions such that f must be a rational function. The full results of this work are detailed in section 4. Note that these results are separate from Theorem

A, and I now have several theorems relating to homogeneous differential polynomials. I will be presenting these (or other) results at the One Day Function Theory Conference in September. I will be submitting these results for publication, and so will hopefully have three accepted papers by this time next year.

Upon completion of the work on non-linear homogeneous differential polynomials, I started looking at composite functions, reading papers by Fletcher, Langley [10] and Lingham [22], referencing papers by Langley, Zheng, Eremenko and Rossi [8][20][23]. The paper with Lingham proved the following result:

Proposition [22]

Let k be a non-negative integer, g a non-constant polynomial, f transcendental entire and Q a rational function. Define F and H as

$$F = f \circ g, \qquad H = F^{(k)} - Q,$$

and suppose that

$$\lim_{r \to \infty} \frac{T(r,f)}{r} = 0.$$

Then the following conclusions hold:

- *If f has finitely many poles then the exponent of convergence of the zeros of H is equal to the order $\rho(F)$ of F.*
- *If $k \geq 2$ and Q is a polynomial, or if $k = 1$ and $Q \equiv 0$, then H has infinitely many zeros.*

I looked into the possibility of extending the first result to transcendental g and Q. Whilst it was easy to show that Q need not be rational, extending it further got extremely messy, and this work is now on a back burner.

My current work is looking at integer-valued functions, that is functions which take integer values on some subset of the natural numbers of non-zero density. My investigation into this began after a talk Langley gave at the first Frontiers of Nevanlinna Theory conference at University College London in late March. A theorem of Pólya [27] states that 2^z is the slowest-growing integer-valued transcendental function, and it occurred to me that the results given at the talk were only concerned with the functions having integer values, they didn't say anything about what those values might actually be. After discussion with Langley, we conjectured that the slowest-growing transcendental function which takes values which are square numbers at integer points is 4^z, but the papers I have looked at [9][19] do not have any stage at which it would be possible to insert the square number requirement, and so for now this remains just a conjecture. I however continued looking at integer-valued functions, particularly the following result by Fletcher and Langley:

Proposition [9]

Let d, J, λ satisfy

$$0 < d < 1, \quad J \in \mathbb{N}, \quad \lambda > 0, \quad 16\left(\frac{1+\log(1+J/2)}{J}\right) + 8(J-1)\lambda < d^2.$$

Let $E \subset \mathbb{N}$ have lower linear density

$$\underline{D}(E) = \liminf_{n\to\infty} \frac{|E \cap \{1,\ldots,n\}|}{n} > d$$

where $|X|$ denotes the number of elements in the set X. Let the function f be analytic of exponential type less than λ in the closed right half plane Ω, and assume that $f(n) \in \mathbb{Z}$ for every $n \in E$. Then f is a polynomial.

It appears that this result may be generalised to meromorphic functions in the plane using the Poisson-Jensen formula. Good progress has been made, and I am hopeful that it will be concluded with a definitive answer.

3 Research and thesis plan

I intend to continue my work on integer-valued functions for now, and with various holidays I expect it to be resolved by the end of July or middle of August, and a short writeup period coincident with possibly trying to extend the result to functions in a half plane rather than the whole plane. I have written up my results on non-linear homogeneous differential polynomials, and will submit them for publication soon. I do not have any specific plans as to what direction to take my research in after that, I may well go back and look for more information on integer-valued functions to see whether my conjecture can be proven.

As regards writing up my thesis, most of that is happening as I go along, writing up results I have proved for publication, to be later combined into the finished document. There will need to be some changes, mainly to the numbering, and also in some cases an introductory section, for instance on the various terms used in the analysis of differential polynomials. I aim to write a general Introduction to Nevanlinna Theory chapter by around Christmas 2011, at which point I will have four or five chapters more or less completed. I would expect to start combining them in summer 2012, for submission between March and August 2013.

4 Draft of a thesis chapter - Non-linear homogeneous differential polynomials

In this chapter, we apply lemmas of Mues and Steinmetz from [24] to non-linear homogeneous differential polynomials in f and $f^{(k)}$ with coefficients which are $O(\log r) + o\left(T(r, f)\right)$ in order to find sufficient conditions such that f is of the form Re^P where R is a rational function and P is a polynomial.

4.1 Introduction

We consider non-linear homogeneous differential polynomials F in f and $f^{(k)}$ with restrictions on the frequency of the zeros, and from there attempt to determine the form of f. We use the standard notation of [16] throughout, and we write $\lambda(r, h)$ for any term which is $O(\log r) + o\left(T(r, h)\right)$ outside some set of finite measure.

Let f be a transcendental meromorphic function. We define

$$u = \frac{f}{f^{(k)}} \tag{4.1}$$

for some $k \geq 1$. Further, let

$$F = f^n + \sum_{j=0}^{n-2} c_j f^j \left(f^{(k)}\right)^{n-j}, \tag{4.2}$$

be a homogeneous non-linear differential polynomial in f and $f^{(k)}$, with coefficients c_j such that $T(r, c_j) = \lambda(r, u)$. We may further rewrite (4.2) as

$$F = \left(f^{(k)}\right)^n \psi$$

where

$$\psi = u^n + \sum_{j=0}^{n-2} c_j u^j. \tag{4.3}$$

We assume that $\overline{N}(r, 1/\psi) = \lambda(r, u)$.

4.2 Lemmata

We begin by stating some useful lemmas, assuming throughout this section that ψ is as in (4.3), that $\overline{N}(r, 1/\psi) = \lambda(r, u)$, and that there is no constant c such that $\psi \equiv cu^n$. We first prove two lemmas from [7], which provide an important step in our working.

Lemma 1 [7]

Suppose $Q[h]$ is a polynomial of degree at most n in a meromorphic function h and its derivatives, with meromorphic functions c satisfying $m(r, c) = \lambda(r, h)$ as coefficients. Then

$$\frac{1}{2\pi} \int_{\left|h\left(re^{i\theta}\right)\right|>1} \log^+ \left|h^{-n} Q[h]\right| d\theta = \lambda(r, h).$$

Proof:

$Q[h]$ is a sum of terms $t = ch^{a_0}(h')^{a_1} \ldots (h^{(p)})^{a_p}$ where $m(r, c) \leq \lambda(r, h)$, and the a_j are non-negative integers whose sum is $N \leq n$. When $|h| > 1$, we have

$$\begin{aligned} |h^{-n}t| &= |c||h'/h|^{a_1} \ldots |h^{(p)}/h|^{a_p} |h|^{N-n} \\ &\leq |c||h'/h|^{a_1} \ldots |h^{(p)}/h|^{a_p} \\ &= T. \end{aligned}$$

We now use the Lemma of the Logarithmic Derivative [16] to obtain,

$$\begin{aligned} \frac{1}{2\pi} \int_{\left|h\left(re^{i\theta}\right)\right|>1} \log^+ \left|h^{-n} Q[h]\right| d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \sum T d\theta \\ &\leq \sum \frac{1}{2\pi} \int_0^{2\pi} \log^+ T d\theta + O(1) \\ &= \sum m(r, T) + O(1) \\ &\leq \sum \left(a_1 m(r, h'/h) + \ldots + a_p m(r, h^{(p)}/h)\right) + \lambda(r, h) \\ &= \lambda(r, h). \end{aligned}$$

QED

Lemma 2 - Clunie's Lemma [7]

Suppose that $h^n P[h] = Q[h]$*, where* h *is meromorphic and* $P[h]$ *and* $Q[h]$ *are polynomials in* h *and its derivatives with meromorphic functions* c *satisfying* $m(r, c) = \lambda(r, h)$ *as coefficients,* $Q[h]$ *being of degree* n *at most. Then,*

$$m(r, P[h]) = \lambda(r, h). \tag{4.4}$$

Proof:

As with $Q[h]$ in Lemma 1, $P[h]$ is a sum of terms of the form $s = ch^{b_0}(h')^{b_1} \dots (h^{(q)})^{b_q}$, where $m(r, c) \leq \lambda(r, h)$, and the b_j are non-negative integers. When $|h| \leq 1$, we have $|s| \leq |c||h'/h|^{b_1} \dots |h^{(q)}/h|^{b_q} = S$, and so

$$\begin{aligned} \frac{1}{2\pi} \int_{\left|h\left(re^{i\theta}\right)\right| \leq 1} \log^+ |P[h]| \, d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \sum S d\theta \\ &\leq \sum \frac{1}{2\pi} \int_0^{2\pi} \log^+ S d\theta + O(1) \\ &= \lambda(r, h). \end{aligned}$$

Further, since $h^n P[h] = Q[h]$, we have by, Lemma 1,

$$\frac{1}{2\pi} \int_{\left|h\left(re^{i\theta}\right)\right| > 1} \log^+ |P[h]| \, d\theta = \frac{1}{2\pi} \int_{\left|h\left(re^{i\theta}\right)\right| > 1} \log^+ \left|h^{-n} Q[h]\right| d\theta = \lambda(r, h),$$

and thus by summation, $m(r, P[h]) = \lambda(r, h)$.

QED

We now move on to several lemmas from [24], which provide the main thrust of our argument by estimating the Nevanlinna functionals of u and $1/u$.

Lemma 3 [24]

With the assumptions of this section on ψ *and* u*, there exists a meromorphic function* $a(z) \not\equiv 0$ *such that*

$$T(r, a) = \lambda(r, u), \tag{4.5}$$

and

$$u' = a + \frac{1}{n}\frac{\psi'}{\psi}u.$$

Remark: This is Lemma 1 of [24], but with the proof slightly modified to take account of our assumptions on the coefficients of u^j.

Proof:

Since $\psi \not\equiv u^n$, we have by (4.3)

$$\psi = u^n + R[u], \tag{4.6}$$

where $R[u] \not\equiv 0$. Differentiating, we have

$$u^{n-1}P[u] = Q[u] \tag{4.7}$$

where P and Q are given by

$$P[u] = nu' - \frac{\psi'}{\psi}u \qquad \text{and} \qquad Q[u] = \frac{\psi'}{\psi}R[u] - R[u]',$$

and the degree of Q is at most $n-2$. We define

$$a = u' - \frac{1}{n}\frac{\psi'}{\psi}u = \frac{P[u]}{n}, \tag{4.8}$$

and apply Clunie's Lemma to (4.7). This gives us that

$$m(r, P[u]) = m(r, na) = m(r, a) + O(1) = \lambda(r, u). \tag{4.9}$$

Suppose $a \equiv 0$, we would have $\psi = cu^n$ for some constant c, which contradicts our assumption of there being no such c. Hence $a \not\equiv 0$.

Now, let z_0 be a pole of a of multiplicity μ. If $u(z_0) \neq \infty$, then $\mu = 1$ and either $\psi(z_0) = 0$, or $\psi(z_0) = \infty$ and thus there must be some j such that c_j has a pole at z_0. If however u has a pole at z_0 of multiplicity ν, we may assume the c_j have poles of order at most η, and by (4.7),

$$(n-1)\nu + \mu \leq 1 + (n-2)\nu + \eta, \qquad \text{and so} \qquad \mu \leq 1 - \nu + \eta \leq \eta, \tag{4.10}$$

where the terms in the first inequality come from u^{n-1}, a, ψ'/ψ, the powers of u in $R[u]$, and the c_j respectively. Hence,

$$N(r,a) \leq \overline{N}\left(r,\frac{1}{\psi}\right) + \sum_{j=0}^{n-2} N(r,c_j) \leq \lambda(r,u) \tag{4.11}$$

by our hypothesis on the zeros of ψ and the characteristics of the coefficients c_j. Thus, combining this with (4.9), we return (4.5).

QED

Lemma 4 [24]

With the assumptions of this section on ψ and u, we have

$$m(r,u) = \lambda(r,u), \tag{4.12}$$

$$m(r,1/u) = \lambda(r,u), \tag{4.13}$$

$$N_1(r,u) = \lambda(r,u), \tag{4.14}$$

$$N_1(r,1/u) = \lambda(r,u), \tag{4.15}$$

where $N_1(r,u) = N(r,u) - \overline{N}(r,u)$, and thus may be considered to only count multiple poles of u. We note here that if u is transcendental, then the $\lambda(r,u)$ reduces to $S(r,u)$ in each equation, and that if u is rational it reduces to $O(\log r)$.

It should be noted that this lemma is as in [24], but with the $S(r,u)$ term expanded out.

Proof:

Recall (4.7), and divide through by $na = P[u]$, yielding $u^{n-2}u = Q[u]/na$, to which we apply Clunie's Lemma, and so (4.12) follows.

Now, divide (4.8) by au, hence

$$\begin{aligned} m\left(r,\frac{1}{u}\right) &= m\left(r,\frac{1}{a}\left(\frac{u'}{u}-\frac{1}{n}\frac{\psi'}{\psi}\right)\right) \\ &\leq T(r,a) + m\left(r,\frac{u'}{u}\right) + m\left(r,\frac{\psi'}{\psi}\right) + O(1). \end{aligned}$$

But, by the Lemma of the Logarithmic Derivative [16], $m(r, u'/u) = O(\log r + \log^+ T(r,u)) = \lambda(r,u)$ outside a set of finite measure. Further, since $T(r,\psi) = O\left(T(r,u) + \sum T(r,c_j)\right)$, we have $m(r,\psi'/\psi) = \lambda(r,u)$. Thus, by (4.5), (4.13) follows.

Let z_0 be a pole of u of order $\nu \geq 2$, and suppose the c_j have poles at z_0 of order at most η, and that at z_0, a has a pole of order $\mu > 0$ or has a zero of order $-\mu \geq 0$. Thus, (4.7) gives (4.10) again, and we have that $\nu - 1 \leq \eta - \mu \leq \eta + \max\{0, -\mu\}$, and so by (4.5)

$$\begin{aligned} N_1(r,u) &\leq N\left(r, \frac{1}{a}\right) + \sum_{j=0}^{n-2} N(r, c_j) \\ &\leq T(r,a) + \sum_{j=0}^{n-2} T(r, c_j) \\ &= \lambda(r,u), \end{aligned}$$

thus proving (4.14).

Finally, suppose z_0 is a zero of u of order $\nu \geq 2$. Then a has a zero of multiplicity at least $\nu - 1$ at z_0, and so by (4.5)

$$N_1(r,u) \leq N\left(r, \frac{1}{a}\right) \leq T(r,a) + O(1) = \lambda(r,u).$$

QED

Lemma 5

For any meromorphic function h, we have

$$N^{2+}(r,h) \leq 2N_1(r,h),$$

where $N^{2+}(r,h)$ counts only multiple poles of h, each according to multiplicity.

Proof:

If z_0 is a pole of h of multiplicity $j > 0$, then $N_1(r,h)$ effectively counts it as a pole of multiplicity $j-1$, and $2N_1(r,h)$ effectively counts it as a pole of multiplicity $2(j-1)$. Since $2(j-1) \geq j \; \forall \, j \geq 2$, we get $N^{2+}(r,h) \leq 2N_1(r,h)$.

QED

Lemma 6 [29]

If $v = f/f'$ is a rational function, then $f = Re^P$, where R is a rational function and P is a polynomial.

Proof:

Since v is rational, then $1/v = f'/f$ must also be rational. This has only simple poles, occuring at the zeros and poles of f. Thus, we can write $1/v$ in the form

$$\frac{1}{v} = \frac{f'}{f} = \sum_{s=1}^{t} \left(\frac{a_s}{z - b_s} \right) + Q(z), \tag{4.16}$$

where the b_s are the zeros and poles of f, $Q(z)$ is a polynomial, and $a_s \in \mathbb{Z}$. Now, in a simply connected region D avoiding the b_s,

$$\begin{aligned} \int \frac{f'}{f} dz = \log f &= \sum_{s=1}^{t} a_s \log(z - b_s) + P(z) \\ &= \log \left(\prod_{s=1}^{t} (z - b_s)^{a_s} \right) + P(z) \end{aligned}$$

where $P(z) = \int Q(z) dz$ is a polynomial. Therefore,

$$\begin{aligned} f(z) &= \exp \left(\log \left(\prod_{s=1}^{t} (z - b_s)^{a_s} \right) + P(z) \right) \\ &= \prod_{s=1}^{t} (z - b_s)^{a_s} e^{P(z)} \\ &= R(z) e^{P(z)}, \end{aligned}$$

where R is a rational function, and neither R nor P depends on D.

QED

Our next lemma extends the previous lemma to the quotient $f/f^{(k)}$, subject to conditions on the frequency of zeros of either numerator or denominator.

Lemma 7

If $u = f/f^{(k)}$ is a rational function, and either f or $f^{(k)}$ have only finitely many zeros, then $f = Re^P$, where R is a rational function and P is a polynomial.

Proof:

By Lemma 6, it is sufficient to prove that $1/v = f'/f$ is a rational function. From the definition of u, we have $f = uf^{(k)}$, and since u is rational, the hypotheses imply that f has only finitely many zeros. Moreover, f has only finitely many poles since a pole of f is a zero of u. Hence,

$$\begin{aligned} N\left(r, \frac{1}{v}\right) = N\left(r, \frac{f'}{f}\right) &= \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) \\ &= O(\log r). \end{aligned} \tag{4.17}$$

Using Lemma 3.5 from [16], we may write

$$\frac{1}{u} = \left(\frac{1}{v}\right)^k + S\left[\frac{1}{v}\right],$$

where S is a differential polynomial in $1/v$ with constant coefficients, of degree at most $k-1$. We rewrite this as

$$\left(\frac{1}{v}\right)^{k-1} \frac{1}{v} = \frac{1}{u} - S\left[\frac{1}{v}\right],$$

and since u is rational we have $T(r, u) = T(r, 1/u) + O(1) = \lambda(r, 1/v) = \lambda(r, v)$ by the First Fundamental Theorem. Thus, Clunie's Lemma implies

$$m\left(r, \frac{1}{v}\right) = \lambda(r, v),$$

and so, using (4.17),

$$T\left(r, \frac{1}{v}\right) = \lambda(r, v) = O(\log r) + o\left(T(r, v)\right),$$

and hence $1/v$ is rational.

QED

Lemma 8

Suppose that h is a meromorphic function and that

$$h^m + d_{m-1}h^{m-1} + \ldots + d_1 h + d_0 \equiv 0$$

where the coefficients d_j are meromorphic functions such that $T(r, d_j) = \lambda(r, h)$. Then h is a rational function.

Proof:

We write

$$-h = d_{m-1} + \ldots + \frac{d_1}{h^{m-1}} + \frac{d_0}{h^{m-1}}$$

to get that

$$N(r, h) \le \sum_{j=0}^{m-1} N(r, d_j), \qquad m(r, h) \le \sum_{j=0}^{m-1} m(r, d_j) + O(1),$$

and thus $T(r, h) = \lambda(r, h)$ and so h is rational.

QED

4.3 Results

Theorem 1

Let u be as in (4.1) with $k \ge 2$, and let ψ be as in (4.3). Suppose that $\overline{N}(r, 1/f) + \overline{N}(r, 1/\psi) = \lambda(r, u)$, and that there is at least one j such that $c_j \not\equiv 0$. Then $f = Re^P$, where R is a rational function and P a polynomial.

Proof:

Suppose that u is rational. Then $\lambda(r, u) = O(\log r)$, and thus f has only finitely many zeros. Hence by Lemma 7, $f = Re^P$.

Now suppose that u is transcendental. Then by Lemma 8, there exists no $c \in \mathbb{C}$ such that $\psi \equiv cu^n$. Using the First Fundamental Theorem of Nevanlinna Theory [16],

$$\begin{aligned} T(r,u) &= T\left(r, \frac{1}{u}\right) + O(1) \\ &= N\left(r, \frac{1}{u}\right) + m\left(r, \frac{1}{u}\right) + O(1) \\ &= N^1\left(r, \frac{1}{u}\right) + N^{2+}\left(r, \frac{1}{u}\right) + m\left(r, \frac{1}{u}\right) + O(1). \end{aligned} \tag{4.18}$$

Hence by (4.13), (4.15) and Lemma 5, we have

$$N^{2+}\left(r, \frac{1}{u}\right) + m\left(r, \frac{1}{u}\right) \leq 2N_1\left(r, \frac{1}{u}\right) + \lambda(r,u) \leq \lambda(r,u).$$

Since for u to have a simple zero, f must have a zero,

$$N^1\left(r, \frac{1}{u}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) = \lambda(r,u).$$

Thus (4.18) gives that $T(r,u) = \lambda(r,u)$, implying that u is rational, a contradiction.

QED

Theorem 2

Let u be as in (4.1) with $k \geq 1$, and let ψ be as in (4.3). Suppose that $\overline{N}(r, 1/f^{(k)}) + \overline{N}(r, 1/\psi) = \lambda(r,u)$, and that there is at least one j such that $c_j \not\equiv 0$. Then $f = Re^P$, where R is a rational function and P a polynomial.

Proof:

Suppose that u is rational. Then $\lambda(r,u) = O(\log r)$, and thus $f^{(k)}$ has only finitely many zeros. Hence by Lemma 7, $f = Re^P$.

Now suppose that u is transcendental. Then by Lemma 8, there exists no $c \in \mathbb{C}$ such that $\psi \equiv cu^n$. Thus by (4.12), (4.14) and Lemma 5,

$$N^{2+}(r,u) + m(r,u) \leq 2N_1(r,u) + \lambda(r,u) = \lambda(r,u).$$

Now, a simple pole of u cannot be a pole of f, and so must be a zero of $f^{(k)}$. Hence,

$$T(r,u) \leq \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + N^{2+}(r,u) + m(r,u) \leq \lambda(r,u),$$

and so u is rational, a contradiction.

QED

4.4 A second look

We now consider a new method in an attempt to apply Theorem 1 for the case $k = 1$, and find a much stronger result by applying a different condition. We first require a lemma.

Lemma 9

Suppose that the transcendental meromorphic function f has a value $\alpha \in \mathbb{C}\backslash\{0\}$ such that

$$\delta = \delta(\alpha, f) = 1 - \limsup_{r\to\infty} \frac{N\left(r, 1/(f-\alpha)\right)}{T(r,f)} > 0. \tag{4.19}$$

Then

$$T(r,f) + T(r,u) = O\left(m(r,u)\right) \qquad \text{(n.e.)}. \tag{4.20}$$

Proof:

We rewrite

$$\frac{1}{f-\alpha} = \frac{f}{f^{(k)}} \frac{f^{(k)}}{f(f-\alpha)} = \frac{f}{\alpha f^{(k)}}\left(\frac{f^{(k)}}{f-\alpha} - \frac{f^{(k)}}{f}\right).$$

By the First Fundamental Theorem, $T(r,f) = T\left(r, 1/(f-\alpha)\right) + O(1)$, and so by (4.19) and the Lemma of the Logarithmic Derivative [16],

$$\begin{aligned}
\left(\delta - o(1)\right) T(r,f) &\leq m\left(r, \frac{1}{f-\alpha}\right) \\
&\leq m\left(r, \frac{f}{f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f-\alpha}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + O(1) \\
&= m\left(r, \frac{f}{f^{(k)}}\right) + o\left(T(r,f)\right) \qquad \text{(n.e.)}
\end{aligned}$$

and so, outside a set of finite measure, $m(r,u) = m(r, f/f^{(k)}) \geq (\delta - o(1))\, T(r,f)$.

However, we also note that

$$T(r,u) = T\left(r, \frac{f}{f^{(k)}}\right) \leq T(r,f) + T(r, f^{(k)}) = O\left(T(r,f)\right) \qquad \text{(n.e.)},$$

and hence

$$(\delta - o(1))\, T(r,f) \leq m(r,u) \leq T(r,u) \leq O\left(T(r,f)\right) \qquad \text{(n.e.)},$$

from which (4.20) follows.

QED

Theorem 3

Let u be as in (4.1) with $k \geq 1$, and let ψ be as in (4.3). Suppose that $\alpha \in \mathbb{C}\backslash\{0\}$ is such that $\delta(\alpha, f) > 0$, that $\overline{N}(r, 1/\psi) = \lambda(r,u)$, and that there is at least one j such that $c_j \not\equiv 0$. Then f is a rational function.

Proof:

Suppose that u is transcendental, then by Lemma 8 ψ/u^n is non-constant and we apply Lemma 4 to give $m(r,u) = \lambda(r,u)$. Thus by Lemma 9, we then have $T(r,u) = \lambda(r,u)$, and so u is not transcendental. Hence assume u is rational, and that f is transcendental. Lemma 9 then gives us $T(r,f) = O\left(m(r,u)\right) = \lambda(r,u) = O(\log r)$, a contradiction. Hence f is rational.

QED

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