Nevanlinna Theory

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Abstract

This dissertation is on the topic of Nevanlinna Theory, a powerful tool from complex analysis. In this report, I begin by studying how Nevanlinna Theory is derived, and continue by showing how its results and methods can be used to solve some interesting problems which are simple to state, but often not to resolve. I end by looking at how Nevanlinna Theory has been used by other authors in recent research.

Contents

1 Introduction

Nevanlinna Theory is a powerful tool from complex analysis, developed by Finnish mathematician Rolf Nevanlinna, and published in 1929. It is especially useful in the analysis of meromorphic functions - functions which are the quotient of two analytic functions - because these functions often have poles, where standard complex analytical techniques such as the Maximum Principle break down. Nevanlinna defined a series of functions which take account of these poles, and which can then in turn be analysed to provide us with properties of the original function.

In this text, we will look at the basic definitions and theory in Section 2, including the First Fundamental Theorem. Section 3 is primarily devoted to the Second Fundamental Theorem, presenting a detailed proof of the theorem and what can be learnt from it, including a generalisation of Picard's Theorem and a look at for which values of n the equation $f^n + g^n = 1$ has solutions. Section 4 deals with Hayman's Alternative, which states that either f takes every finite value infinitely often, or $f^{(t)}$ takes every non-zero finite value infinitely often for any positive integer t - we will prove the theorem, and then give a generalisation, as proved by Bergweiler and Langley [4]. Section 5 starts out by looking at a paper by Bergweiler and Eremenko [3], which is commonly regarded as one of the most important recent papers dealing with Nevanlinna Theory. We will then look at how the results of this paper have been used when dealing with various differential equations, as cited by many notable authors in various publications.

We now begin with the elementary theory.

2 The Elementary Theory

In this section, we will make some basic definitions and statements of syntactic convention, then look at Jensen's formula for the logarithm, before defining the Nevanlinna functionals and looking at some basic results.

2.1 Basic definitons

Before we can do analysis of meromorphic functions, we first need to define precisely what we mean by a meromorphic function. Nevanlinna, in [9], defined it thus:

Definition 2.1.1

A meromorphic function is one which is analytic for all values $z \in \mathbb{C}$ *except for isolated points where poles (points where the function takes the value* ∞*) occur.*

In practice, we can consider a meromorphic function to be the quotient of two analytic functions, having poles when the denominator is zero, and zeros when the numerator is zero. Points where both denominator and numerator are zero can usually be handled using limits. In this manner, $\tan z$ is meromorphic, since it has poles when $\cos z$ takes the value 0, e.g. at $z = \frac{\pi}{2}$. We note here that if $f(z)$ has a pole at some $z = z_0$, then $1/f$ has a zero at that point. It is also important to note that meromorphic functions can have only isolated poles - for instance the function $f = 1/(|z| - 1)$ is not meromorphic as it takes the value of infinity at all points on the unit circle. The reasoning behind this restriction will become apparent later.

Definition 2.1.2

Given an analytic function f *which is not identically* 0*, and a point* $z_0 \in \mathbb{C}$ *such that* $f(z_0) = 0$*, we define the* order *or* multiplicity *of the zero at that point as the least* n *such that in the Taylor expansion of* f *about* z_0 *the coefficient of* $(z - z_0)^n$ *is non-zero.*

We similarly define the multiplicity of a pole of a function $g(z)$ *at a point* $z \in \mathbb{C}$ *as the multiplicity of the zero at that point of* $1/g$*. A pole of multiplicity 1 is called a* simple pole.

It is of note that if we have two functions f and g with zeros (or poles) at some point, of multiplicity j and k respectively, then fg has a zero (or pole) of multiplicity $j+k$. If f has a pole of multiplicity j and g has a zero of multiplicity k at some point, then fg has a zero of multiplicity $\max\{0, k-j\}$ at that point, or a pole of multiplicity $\max\{0, j-k\}$. If $j = k$ then fg takes some non-zero finite value at that point (see Example 3.1.3).

Example 2.1.3

$\sin^2 z$ has a zero of multiplicity 2 at $z = 0$, while $\tan z$ has a simple pole at $z = \frac{\pi}{2}$.

It is clear from the Taylor series that if at some point f has a zero of multiplicity n, then f' has a zero of multiplicity $n-1$ at that point; and that similarly if f has a pole of multiplicity m at some point, then f' has a pole of multiplicity $m+1$ at that point.

We also need to define certain sets.

Definition 2.1.4

We define the open disc $B(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ *as the set of all points within a radius* r *of* a*, and its boundary to be the circle* $S(a,r) = \{z \in \mathbb{C} : |z-a| = r\}$*. We further define* $\overline{B}(a,r) = B(a,r) \cup S(a,r)$*.*

Definition 2.1.5

For our functionals, we need what is known as the positive logarithm*. We define this as* $\log^+ x = \max(\log x, 0)$*, for any* $x > 0$*. It is evident that it satisfies*

$$\log x = \log^+ x - \log^+ \frac{1}{x}.$$

Note that in this text $\log$ *is taken to mean* $\log_e$*.*

We next define the symbols O and o:

Definition 2.1.6

Let $f(r)$ *and* $g(r)$ *be functions defined on* $[a,\infty)$*, with* $f(r)$ *complex valued and* $g(r)$ *real and positive. We say that* $f(r) = O(g(r))$ *as* $r \to \infty$ *if there exist constants* c*,* r_0 *such that* $|f(r)| \leq cg(r)$ $\forall r \geq r_0$*. We say* $f(r) = o(g(r))$ *if* $\frac{f(r)}{g(r)} \to 0$ *as* $r \to \infty$*.*

Example 2.1.7

$\sin r = o(r)$ and $\tanh r = O(1)$ as $r \to \infty$. In particular, if $f(z) = O(1)$, then f is bounded.

We finally need to define the notions of $\limsup$ and $\liminf$.

Definition 2.1.8

Let $f(x)$ *be a real-valued function defined over the real variable* x*. Then we define the following notions:*

$$\limsup_{x\to\infty} f(x) = \lim_{x\to\infty} [\sup\{f(y) : y \geq x\}]$$
$$\liminf_{x\to\infty} f(x) = \lim_{x\to\infty} [\inf\{f(y) : y \geq x\}]$$

Note that $\limsup f(x) = \liminf f(x)$ if and only if $\lim f(x)$ exists.

Now we have these definitions, we can move onto looking at more complex matters.

2.2 Jensen's Formula

In this section, we will describe Jensen's Formula, which will be used in several places throughout this text.

Theorem 2.2.1 - Poisson's formula for the logarithm

Let $0 < r < \infty$*,* $S = \{z \in \mathbb{C} : |z| \leq r\}$*, and a function* f *be meromorphic on some domain containing* S *with neither zeros nor poles in* $B(0, r)$*. Let us label the distinct poles and zeros of* f *on the circle* $S(0, r) := \{z \in \mathbb{C} : |z| = r\}$ *as the points* $\chi_1 \ldots \chi_n$*. Then we may define an analytic branch* U *of* $\log f$ *on a simply connected domain containing* $S \setminus \{\chi_1 \ldots \chi_n\}$*, and, for* $|a| < r$*,*

$$U(a) = \frac{1}{2\pi} \int_0^{2\pi} U\left(Re^{i\theta}\right) \frac{R^2 - |a|^2}{\left|Re^{i\theta} - a\right|^2} d\theta. \tag{2.1}$$

Proof:

We may define such an analytic branch by noting that f is meromorphic in $B(0, R')$ with no zeros or poles in $R < |z| < R'$ since f has only isolated poles and zeros.

Now, let δ be small and positive, and let Γ_δ be the circle $S(0,R)$ described once anticlockwise, except that each χ_j is avoided by describing an arc κ_j of the circle $S(\chi_j,\delta)$ clockwise. We now define the Möbius transformation ω as,

$$\omega(z) = R^2\left(\frac{z+a}{r^2+\overline{a}z}\right).$$

This Möbius transformation maps S onto itself bijectively with boundary mapped onto boundary and an inverse map

$$z = R^2\left(\frac{\omega-a}{R^2-\overline{a}w}\right), \quad \frac{dz}{d\omega} = R^2\left(\frac{R^2-|a|^2}{(R^2-\overline{a}\omega)^2}\right).$$

Let γ_δ be the preimage under ω of the curve Γ_δ - this is another closed curve described once anticlockwise, enclosing the origin but none of the preimages of χ_j. Thus, $U(\omega(z))$ is defined on and inside γ_δ, and using the Cauchy Residue Theorem,

$$U(a) = \frac{1}{2\pi}\int_{\Gamma_\delta} U(\omega)\left(\frac{\omega}{\omega-a}\right)\left(\frac{R^2-|a|^2}{R^2-\overline{a}\omega}\right)\frac{d\omega}{i\omega}. \tag{2.2}$$

But, as $\omega \to \chi_j$ inside $B(0,R)$, there are nonzero constants c_j and integers m_j such that

$$g(\omega) \sim c_j(\omega-\chi_j)^{m_j} \text{ and } U(\omega) = m_j \log\frac{1}{|\omega-\chi_j|} + O(1)$$

In particular, $\arg g(\omega)$ is bounded as $\omega \to \chi_j$ in $B(0,R)$, and (on κ_j for small δ)

$$U(\omega) = O\left(\log\frac{1}{\delta}\right).$$

Hence, as $\delta \to 0$, the contribution to the integral from each κ_j tends to 0, so choosing $\omega = Re^{i\theta}$ gives

$$U(a) = \frac{1}{2\pi}\int_{S(0,R)} U(\omega)\left(\frac{\omega}{\omega-a}\right)\left(\frac{R^2-|a|^2}{R^2-\overline{a}\omega}\right)\frac{d\omega}{i\omega} = \frac{1}{2\pi}\int_0^{2\pi}\frac{U(\omega)(R^2-|a|^2)}{(\omega-a)(\overline{\omega}-\overline{a})}d\theta,$$

from which we regain (2.1).

QED

Theorem 2.2.2 - The Poisson-Jensen formula

Let R be finite and positive, f be meromorphic and not identically 0 in $\overline{B}(0,R)$. Then

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| \frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\phi)} d\theta + d\log\left|\frac{z}{R}\right| +$$

$$+\sum_{j=1}^{m}\log\left|\frac{R(z-a_j)}{R^2-\overline{a_j}z}\right| - \sum_{k=1}^{n}\log\left|\frac{R(z-b_k)}{R^2-\overline{b_k}z}\right|, \qquad (2.3)$$

where $a_1,\ldots,a_m$ and $b_1,\ldots,b_n$ are respectively the zeros and poles of f in $0<|z|<R$, with repetition according to multiplicity.

Proof:

First, assume that near the origin $f(z) = cz^d(1+o(1))$ as $z \to 0$, where $d \in \mathbb{Z}$ and c is a finite non-zero constant. This states that cz^d is the first term of the Laurent expansion of f valid in some annulus $0<|z|<s_0$. Then

$$g(z) = f(z)\frac{R^d}{z^d}\prod_{j=1}^{m}\left(\frac{R(z-a_j)}{R^2-\overline{a_j}z}\right)^{-1}\prod_{k=1}^{n}\left(\frac{R(z-b_k)}{R^2-\overline{b_k}z}\right)$$

is meromorphic on $\overline{B}(0,R)$, and analytic and non-zero in $B(0,R)$. Also, on $|z|=R$, $|f(z)|=|g(z)|$. We take the real part of (2.1), and choose $U=\log|g(z)|$, $z=re^{i\phi}$, with $\phi\in\mathbb{R}$ and $0\le r<R$, then (2.3) follows from

$$U(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} U(Re^{i\theta})\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\phi)}d\theta.$$

QED

Corollary 2.2.3 - Jensen's formula

Let f be meromorphic in $B(0,R)$, where the set of a_j and b_k are the zeros and poles of f respectively in that region, then

$$\log|c| = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(Re^{i\theta})\right| d\theta + \sum_{j=1}^{m}\log\frac{|a_j|}{R} - \sum_{k=1}^{n}\log\frac{|b_k|}{R} - d\log R. \qquad (2.4)$$

This is derived from the limit of (2.3) as $z\to 0$. If $f(0)\neq 0,\infty$, then $c=f(0)$.

2.3 The Nevanlinna functionals

We are now in a position to start defining the main Nevanlinna functionals.

Definition 2.3.1

For a function f meromorphic in the disc $\overline{B}(0,r)$, define the function $n(r,f)$, the unintegrated counting function, as the number of poles of f in $\overline{B}(0,r)$, counting multiplicity.

For instance, $\csc^2 z$ has double poles at $0 \pm k\pi$ for positive integer k. Thus $n(k\pi, \csc^2 z) = 2 + 4k$.

It is here that we understand why we require that meromorphic functions have only isolated poles. Returning to our example of $f = 1/(|z| - 1)$, we see that $n(r,f) = 0$ for $0 \le r < 1$, but is undefined for $r \ge 1$. It is also of note that $n(r,f)$ is stepwise non-decreasing.

Definition 2.3.2 - The Nevanlinna functionals

We first define what is known as the (Integrated) Counting Function, *$N(r,f)$:*

$$N(r,f) = \int_0^r \left[n(t,f) - n(0,f)\right] \frac{dt}{t} + n(0,f) \log r \tag{2.5}$$

If f has, counting multiplicity, p poles on $|z| = q$, then these contribute p to $n(r,f) - n(0,f)$ for $0 < q \le r$, and thus $p \log(r/q)$ to $N(r,f)$.

We also define the Proximity Function, *$m(r,f)$:*

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta \tag{2.6}$$

This function is so-called because it gives the proximity of f to infinity, while $N(r,f)$ counts when f does reach infinity.

Finally, we define the Nevanlinna Characteristic*, $T(r,f)$:*

$$T(r,f) = m(r,f) + N(r,f) \tag{2.7}$$

We note that if f is an entire function it has no poles, and so $N(r,f) \equiv 0$, and therefore $T(r,f) \equiv m(r,f)$.

Let us look at some examples so that we may gain a better understanding of these functionals.

Examples 2.3.3
(i) Let $f = \dfrac{g(z)}{h(z)}$ where g and h are polynomials of degree a and b respectively, with no common zeros and $h \not\equiv 0$. Since h has b roots, for large r we can say that $N(r,f) = b\log r + O(1)$. Also, as $z \to \infty$, $f \sim cz^{a-b}[1+o(1)]$ for some constant $c \neq 0$, thus

$$\log|f(z)| = (a-b)\log z + O(1),$$

from which we deduce that as $r \to \infty$,

$$m(r,f) = \max\{a-b, 0\}\log r + O(1).$$

Thus,

$$T(r,f) = \max\{a-b,0\}\log r + b\log r + O(1) = \max\{a,b\}\log r + O(1).$$

as $r \to \infty$.

(ii) Let $f = e^z$. Since f is entire, $T(r,f) \equiv m(r,f)$.

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| e^{re^{i\theta}} \right| d\theta.$$

Now, $|e^z| = e^{Re(z)}$ for any $z \in \mathbb{C}$. Here, the exponent of e is $re^{i\theta}$, the real part of which is $r\cos\theta$, thus we have

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \max\{r\cos\theta, 0\}d\theta.$$

The function $\cos\theta$ is only positive in the region $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$, or equivalently $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, thus we can write our formula:

$$T(r,f) = \frac{r}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log^{+} e^{\cos\theta} d\theta = \frac{r}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta = \frac{r}{\pi}.$$

(iii) Let f be meromorphic in the plane, with k a positive integer and set $g = f\left(z^k\right)$. We shall examine the relations between the functionals for f and g.

First, $n(r,f)$. Suppose f has a pole of multiplicity c at $z = z_0$. Then g has poles of multiplicity c at the points $z^k = z_0$ - that is k distinct points, unless $z_0 = 0$, in which case g has a pole of multiplicity ck at 0. Thus for every pole of f, g has k poles. Now, suppose that in $n(r_f, f)$, we take $r_f = r_0$, ie $|z| = r_0$, then for g, we have that $r_g = |z|^k = r_0^k$. Thus, when combined, $n(r,g) = kn(r^k, f)$.

Now we consider $N(r,f)$. We start with $N(r,g)$, replacing $n(r,g)$ with $kn(r^k,f)$ as derived above, and then perform the substitution $T = t^k$, with $\frac{dt}{dT} = \left(kt^{k-1}\right)^{-1}$:

$$\begin{aligned}
N(r,g) &= \int_0^r \left[n(t,g) - n(0,g)\right]\frac{dt}{t} + n(0,g)\log r \\
&= \int_0^r \left[kn(t^k,f) - kn(0,f)\right]\frac{dt}{t} + kn(0,f)\log r \\
&= \int_0^{r^k} \left[n(T,f) - n(0,f)\right]\frac{kdT}{t}\frac{1}{kt^{k-1}} + n(0,f)\log r^k \\
&= \int_0^{r^k} \left[n(T,f) - n(0,f)\right]\frac{dT}{T} + n(0,f)\log r^k \\
&= N(r^k, f).
\end{aligned}$$

We come now to $m(r, f)$, and use the substitution $\phi = k\theta$:

$$\begin{aligned} m(r,g) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| g\left(re^{i\theta}\right)\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(r^k e^{ik\theta}\right)\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2k\pi} \log^+ \left| f\left(r^k e^{i\phi}\right)\right| \frac{d\phi}{k} \\ &= \frac{k}{2k\pi}\int_0^{2\pi} \log^+ \left| f\left(r^k e^{i\phi}\right)\right| d\phi \\ &= m(r^k, f). \end{aligned}$$

Note that since $m(r, g) = m(r^k, f)$ and $N(r, g) = N(r^k, f)$, it follows that

$$T(r, g) = T(r, f(z^k)) = T(r^k, f). \tag{2.8}$$

From this it follows that $T(r, e^{z^k}) = r^k/\pi$.

(iv) Let us now analyse $f^k(z)$ similarly. We note that if f has a pole (or zero) of multiplicity n at some point, then f^k must have a pole (or zero) of multiplicity nk at that point. Thus, $n(r, f^k) = kn(r, f)$. It is simple to see that when we use this as a substitution, we find that $N(r, f^k) = kN(r, f)$. Finally, using the fact that $\log^+ |f^k| = k \log^+ |f|$, we find that $m(r, f^k) = km(r, f)$ and thus that

$$T(r, g) = T(r, f^k) = kT(r, f). \tag{2.9}$$

It thus follows that $T(r, e^{kz}) = kr/\pi$.

(v) Now, for some finite non-zero constant a, let us look at $g = f(az)$. It is obvious that $n(r, g) = n(ar, f)$, and that since $n(r, f)$ is a function dependant solely on r and f, it is invariant under rotations and thus is indeed equal to $n(|a|r, f)$. When substituting into $N(r, g)$ using $T = at$ we find:

$$N(r, g) = \int_0^{|a|r} [n(T, f) - n(0, f)] \frac{dT}{T} + n(0, f) \log r.$$

Thus $N(r,g) = N(|a|r,f) - n(0,f)\log|a|$. Note that $m(r,f)$ is also invariant under rotations - we can simply perform a substitution for θ. Using this, we find that $m(r,g) = m(|a|r,f)$, and thus that

$$T(r,g) = T(r,f(az)) = T(|a|r,f) - n(0,f)\log|a|. \tag{2.10}$$

From this, it follows that $T(r,e^{az}) = |a|r/\pi$.

Lemma 2.3.4

Let f and g be meromorphic and non-constant in the plane, then the following hold:

$$\begin{aligned} T(r,fg) &\leq T(r,f) + T(r,g) & (2.11) \\ T(r,f+g) &\leq T(r,f) + T(r,g) + \log 2. & (2.12) \end{aligned}$$

Proof:

We start by looking at the poles of the functions. A pole of fg can only occur at a pole of f or g - let us, without loss of generality, consider a pole of f at the origin. Then if g has a pole, $n(0,fg) = n(0,f) + n(0,g)$. If g has neither a pole nor a zero, then again, $n(0,fg) = n(0,f) + n(0,g)$. If g has a zero, then fg either has no pole, or has a pole of lower multiplicity than f, so

$$n(0,fg) \leq n(0,f) + n(0,g).$$

Thus, for any given pole of fg with multiplicity n, the sum of the multiplicites of the poles at that point of f and g are at least n, and thus $n(r,fg) \leq n(r,f) + n(r,g)$, which yields

$$N(r,fg) \leq N(r,f) + N(r,g)$$

when when substituted into the counting function.

Similarly, if $f+g$ has a pole, then at least one of f and g has a pole, and so looking (without loss of generality) at a pole of $f+g$ at the origin,

$$n(0,f+g)=\max\{n(0,f),n(0,g)\}\leq n(0,f)+n(0,g).$$

Through a simple translation of the origin we can see this holds for any pole, and thus

$$n(r,f+g)\leq n(r,f)+n(r,g),$$

which yields

$$N(r,f+g)\leq N(r,f)+N(r,g)$$

when when substituted into the counting function.

Now we look at the proximity function. Since

$$\log^{+}|fg|=\max\{0,\log|f|+\log|g|\}\leq\log^{+}|f|+\log^{+}|g|,$$

it follows that

$$m(r,fg)\leq m(r,f)+m(r,g),$$

and thus when combined with $N(r,fg)$, (2.11) follows. Also, since

$$\log^{+}|f+g|\leq\log^{+}|2\max\{f,g\}|\leq\log^{+}|f|+\log^{+}|g|+\log 2,$$

it is apparent that

$$m(r,f+g)\leq m(r,f)+m(r,g)+\log 2,$$

and hence when combined with $N(r,f+g)$, (2.12) follows.

QED

Example 2.3.5

Let $f = \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$. By (2.12), we see that

$$T(r, f) \leq T(r, e^{iz}) + T(r, e^{-iz}) + \log 2 = (2r/\pi) + O(1).$$

Now let us check this through more elementary methods - note first that neither e^{iz}, nor $\cos z$ have any poles in $\mathbb{C}$, thus we are concerned solely with the proximity function. Now, since $e^{i(x+iy)} = e^{ix}e^{-y}$, it is clear that in the upper half plane, where $y \geq 0$, the term in e^{-iz} dominates, and hence here $\cos z = e^{-iz}/2 + O(1)$. Similarly, in the lower half plane, e^{iz} dominates. Thus, when combined, we find that

$$T(r, \cos z) = T(r, e^{iz}) + T(r, e^{-iz}) + O(1) = (2r/\pi) + O(1),$$

as shown by (2.12).

Lemma 2.3.6 - Comparison of $T(r, f)$ and $M(r, f)$

Let us define the maximum function in the usual manner,

$$M(r, f) = \max\{|f(z)| : z \in S(0, r)\},$$

and let f have no poles in $B(0, r)$. Then

$$T(r, f) \leq \log^+ M(r, f) \tag{2.13}$$

$$\log M(r, f) \leq \left(\frac{R + r}{R - r}\right) T(R, f). \tag{2.14}$$

Proof:

The first of these inequalities follows from the fact that here $N(r, f) \equiv 0$, and $\log^+ |f(re^{i\theta})| \leq \log^+ M(r, f)$. For the second, we apply the Poisson-Jensen formula (2.3), taking $f(z) = M(r, f)$, $|z| = r$ and $R > r \geq 0$. Since there are no poles, the term in b_k vanishes. Since $|z| < R$, the term in $\left|\frac{z}{R}\right|$ is negative, as is the term in a_j. Hence,

$$\log M(r, f) \leq \frac{1}{2\pi} \int_0^{2\pi} \log\left|f(Re^{i\theta})\right| \frac{(R - r)(R + r)}{R^2 + r^2 - 2Rr\cos(\theta - \phi)} d\theta.$$

But, $R^2 + r^2 - 2Rr\cos(\theta - \phi) \geq R^2 + r^2 - 2Rr = (R-r)^2$, thus,

$$\log M(r,f) \leq \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\theta})\right| \frac{R+r}{R-r} d\theta,$$

and using $\log z \leq \log^+ z$, the inequality follows.

QED

Lemma 2.3.7

If $0 < r < R$, *then*

$$N(R,f) \geq n(r,f)\log\frac{R}{r} + n(0,f)\log r. \tag{2.15}$$

Proof:

$$\begin{aligned} N(R,f) &= \int_0^R [n(t,f) - n(0,f)]\frac{dt}{t} + n(0,f)\log R \\ &\geq \int_r^R [n(t,f) - n(0,f)]\frac{dt}{t} + n(0,f)\log R \\ &\geq \int_r^R [n(r,f) - n(0,f)]\frac{dt}{t} + n(0,f)\log R \\ &= [n(r,f) - n(0,f)]\log\frac{R}{r} + n(0,f)\log R, \end{aligned}$$

from which the result follows.

QED

Definition 2.3.8 - The order of a meromorphic function

Let f *be meromorphic in the plane. Then we define the* order $\rho(f)$ *by*

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

We say that f *has* finite order *if* $\rho(f) < \infty$, *or equivalently* $T(r,f) = O(r)$.

We saw earlier that $T(r, e^z) = r/\pi$, thus the order of e^z is 1. It is similarly apparently that the order of e^{z^n} is n.

Lemma 2.3.9

Let f be meromorphic in $\mathbb{C}$ and not a rational function. Then, as $r \to \infty$,

$$\frac{T(r,f)}{\log r} \to \infty \tag{2.16}$$

Furthermore, the order of a rational function is zero.

Proof:

Note that we saw in Example 2.3.3 (i) that if f is rational, then $T(r,f) = O(\log r)$. We now prove the converse. Let f be meromorphic and non-constant in $\mathbb{C}$, and let $(r_n) \to \infty$ be some sequence, on which $T(r_n, f) = O(\log r_n)$. Then by (2.15), where $t_n^2 = r_n$ for large r_n,

$$n(t_n, f) \log t_n \leq N(t_n^2, f) \leq T(t_n^2, f) \leq a \log t_n,$$

for come finite constant a, and thus f has finitely many poles. Thus we can define a polynomial P such that $g = Pf$ is entire - i.e. the zeros of P coincide with the poles of f - and $T(r_n, g) = O(\log r_n)$. Now, using (2.14) and $r_n = 2s_n$,

$$\log M(s_n, g) \leq 3T(r_n, g) \leq b \log r_n,$$

for some finite constant b, and hence $M(s_n, g) \leq r_n^b$, i.e. there exists $k \in \mathbb{Z}_{>0}$ such that $|g(z)| \leq (s_n)^k$ on the circles $|s_n| \to \infty$. Thus, using the Cauchy Integral Formula, we find that $g^{(k)}$ is bounded, and thus constant by Liouville's Theorem, which shows us that g is a polynomial. Hence $f = g/P$ is the quotient of two polynomials, and thus it rational, and so $T(r,f) = O(\log r)$ if and only if f is rational. The second part of the lemma follows since if f is rational,

$$\rho(f) = \frac{\log^+ T(r,f)}{\log r} = \frac{\log^+ k \log r}{\log r} \to 0$$

for some constant k as $r \to \infty$.

QED

2.4 The First Fundamental Theorem

In this section we will now look at the First Fundamental Theorem, which deals with equidistribution of functions. To do this we need to rewrite Jensen's Formula.

Lemma 2.4.1

Jensen's Formula, (2.4), can be rewritten as

$$\begin{aligned} \log|c| &= T(R,f) - T\left(R,\frac{1}{f}\right) && (2.17) \\ &= m(R,f) + N(R,f) - m\left(R,\frac{1}{f}\right) - N\left(R,\frac{1}{f}\right) && (2.18) \end{aligned}$$

where $c = f(0)$ *if* $f(0) \neq 0, \infty$.

Proof:

We recall that $\log x = \log^+ x - \log^+ x^{-1}$, and thus

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| d\theta = m(R,f) - m\left(R,\frac{1}{f}\right). \qquad (2.19)$$

It can be proved (see [8]) that

$$-\sum_{k=1}^{n} \log\frac{|b_k|}{R} = \sum_{k=1}^{n}\log\frac{R}{|b_k|} = \int_0^R [n(t,f) - n(0,f)]\frac{dt}{t} = N(R,f) - n(0,f)\log R.$$

Since $f(0) \sim cz^d$, $n(0,f) - n(0,1/f) = -d$, and so

$$\sum_{j=1}^{m}\log\frac{|a_k|}{R} - \sum_{k=1}^{n}\log\frac{|b_k|}{R} - d\log R = N(R,f) - N\left(R,\frac{1}{f}\right),$$

which when combined with (2.19) yields the result. Modifications can be made if $f(0) \in \{0,\infty\}$, and these can be found in [6].

QED

This lemma allows us to prove the main theorem of this chapter.

Theorem 2.4.2 - The First Fundamental Theorem

Let f be a non-constant meromorphic function, and $a \in \mathbb{C}$. Then

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1). \tag{2.20}$$

Proof:

By Lemma 2.4.1,

$$T\left(r, \frac{1}{f-a}\right) = T(r, f-a) + O(1)$$

for some finite constant a. Now, by (2.12),

$$\begin{aligned} T(r, f-a) &\leq T(r,f) + O(1) \\ &= T(r, (f-a)+a) \leq T(r, (f-a)) + O(1), \end{aligned}$$

and so

$$T(r, f-a) = T(r,f) + O(1).$$

Combining these two gives us the result.

QED

For brevity, we will sometimes write $T\left(r, \dfrac{1}{f-a}\right)$ as $T(r,a,f)$, and the m and N functions similarly. Note that $T(r,f) \equiv T(r,\infty,f)$.

Examples 2.4.3

(i) Let f be non-constant and meromorphic in $\mathbb{C}$. We have seen previously (Example 2.3.3 (i), Lemma 2.3.9) that as $r \to \infty$, $T(r,f) \to \infty$. Thus, using the First Fundamental Theorem, in $T\left(r, 1/(f-a)\right)$, either f takes the value a very often so that N is large, or f is close to a somewhere on the circle $|z| = r$.

(ii) Let $f = e^z$. Then, as we have seen before, $m(r, f) = m(r, 0, f) = \dfrac{r}{\pi}$, and $N(r, f) = N(r, 0, f) = 0$, thus $T(r, f) = T(r, 0, f)$. If $a \neq 0$, then we can note that $m(r, a, f)$ is small, but, by either Picard's Theorem or direct computation, f has lots of a-points, thus $N(r, a, f)$ is large.

(iii) Let S be a Möbius transformation. Then, using (2.11), (2.12), and the First Fundamental Theorem,

$$\begin{aligned} T(r, S(f)) = T\left(r, \frac{af+b}{cf+d}\right) = T\left(r, a' + \frac{b'}{f+d'}\right) &\leq T\left(r, \frac{1}{f+d'}\right) + O(1) \\ &= T(r, f) + O(1). \end{aligned}$$

Now, we can rewrite $\tan z$ as a Möbius transformation of e^{2iz}. Using (2.10), we see that $T(r, e^{2iz}) = T(2r, e^z) = 2r/\pi$, and thus $T(r, \tan z) = (2r/\pi) + O(1)$.

(iv) Let $P(z)$ be a polynomial of degree k, $P = a_k z^k + \ldots$. It is clear that e^P is entire, so $T(r, e^P) \equiv m(r, e^P)$. Now,

$$\log^+ \left|e^P\right| \leq \sum_{j=0}^{k} \log^+ \left|e^{a_j z^j}\right|,$$

and so

$$T\left(r, e^P\right) \leq \sum_{j=0}^{k} T\left(r, e^{a_j z^j}\right).$$

However, we know that $T(r, e^{a_j z^j}) = |a_j| r^j / \pi = O(r^j)$, thus

$$T\left(r, e^P\right) \leq |a_k| r^k / \pi + O(r^{k-1}).$$

3 The Second Fundamental Theorem

In this section we shall look at the Second Fundamental Theorem and some of its applications, including a proof of Picard's Theorem and a variant of Fermat's Last Theorem for functions. We begin by showing that $T(r,f)$ is non-decreasing.

3.1 Cartan's Identity and the Logarithmic Derivative

Theorem 3.1.1 - Cartan's Identity

Let $f(z)$ be meromorphic in $B(0,R)$, with $0<r<R$ and $f(0)\neq\infty$. Then

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}N\left(r,e^{i\theta},f\right)d\theta+\log^+|f(0)| \tag{3.1}$$

In particular, $T(r,f)$ is non-decreasing.

Proof:

We use Jensen's Formula (2.4) with $f=\mu-z$ and $R=1$ for some finite constant μ, noting that f has no poles. If $|\mu|\geq 1$, then there are no zeros to deal with, and the formula shows that the integral is equal to $\log|\mu|$. If however $|\mu|<1$, then the term dealing with the zeros of f, $\log(|a_j|/R)$ is equal to $\log|\mu|$ as there is only a single zero, when $z=\mu$. Thus the integral is equal to 0. Thus we see that

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\mu-e^{i\phi}\right|d\phi=\log^+|\mu|.$$

We again apply Jensen's formula, this time to $f(z)-e^{i\phi}$, yielding

$$\log\left|f(0)-e^{i\phi}\right|=\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(re^{i\theta}\right)-e^{i\phi}\right|d\theta+N(r,f)-N(r,e^{i\phi},f).$$

We now integrate both sides with respect to ϕ, and then change the order of integration for the double integral. This is allowable since the equation is absolutely convergent. For a detailed proof of this, see [8], pages 23-24. Hence,

$$\log^+|f(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta+N(r,f)-\frac{1}{2\pi}\int_0^{2\pi}N\left(r,e^{i\phi},f\right)d\phi,$$

from which the identity is evident as the first term on the right is $m(r, f)$. That $T(r, f)$ is non-decreasing as $N(r, a, f)$ has this property.

QED

Lemma 3.1.2 - Lemma of the Logarithmic Derivative

Let f be non-constant and meromorphic in the plane. Then there are positive constants c_j such that as $r \to \infty$ outside some set of finite measure,

$$m\left(r, \frac{f'}{f}\right) \leq c_1 \log r + c_2 \log T(r, f). \tag{3.2}$$

If f is of finite order, then $m(r, f'/f) = O(\log r)$.

For a proof, see [8], page 27. We note that this estimate is necessary only for transcendental functions, as for any rational function, $f'/f \to 0$ as $r \to \infty$.

We shall now look at an example which uses the Lemma of the Logarithmic Derivative.

Example 3.1.3

Let f be an entire function of finite order, a_1 and a_2 be two distinct complex numbers, and let $h(z)$ be defined as

$$h(z) = \frac{[f'(z)]^2}{(f(z) - a_1)\,(f(z) - a_2)}.$$

We will show that if all a_1- and a_2-points of f have multiplicity ≥ 2, then h is a polynomial.

First, if f has an a_j-point of multiplicity k at some point, then f' has a zero of multiplicity $k-1$ at that point. Since the a_j are distinct, we can, without loss of generality, consider only what happens if $f(z_0) = a_1$. At this point, $f - a_1$ has a zero of multiplicity $k \geq 1$, and $(f')^2$ has a zero of multiplicity $2(k-1)$, and so

h has a zero of multiplicity $2(k-1)-k = k-2$. Therefore, if $k \geq 2$, h does not have a pole, and thus if $k \geq 2$ for a_j-points of f, h is entire.

Now, since h is entire, $T(r,h) \equiv m(r,h)$. Let $h_j = f'/(f-a_j)$, such that $h = h_1 h_2$. By the Lemma of the Logarithmic Derivative (3.2), $m(r,h_j) = O(\log r)$, as f is of finite order, and so, by the properties of $\log^+$,

$$\begin{aligned} m(r,h) = m(r,h_1h_2) &\leq m(r,h_1) + m(r,h_2) \\ &= O(\log r). \end{aligned}$$

We now apply Lemma 2.3.9, and see that this implies that h is rational, and so $h = \frac{P(z)}{Q(z)}$ where P and Q are polynomials. However, since h is also entire, h has no poles, and so any roots of Q are cancelled out by those of P, and so h is a polynomial.

Let us now try and find an example of such a function. A simple, though trivial, example is $f = \sin^2 z$, $a_1 = 0$, $a_2 = 1$. It is fairly simple to prove that all zeros of g are of multiplicity 2, and thus,

$$h = \frac{(2\sin z \cos z)^2}{\sin^2 z(\sin^2 z - 1)} = \frac{4\sin^2 z \cos^2 z}{\sin^2 z(-\cos^2 z)} = -4,$$

which is clearly entire.

Definition 3.1.4

We write $S(r,f)$ for any term $O(\log^+(rT(r,f)))$ outside some set of finite measure. If f is not rational, then $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside some set of finite measure. If f has finite order, then $S(r,f) = O(\log r)$.

Note that $m(r,f'/f) = S(r,f)$.

Definition 3.1.5

We let $\overline{n}(r, f)$ *count the number of poles of* f *within* $\overline{B}(0, r)$ *without regards to multiplicity, and define* $\overline{N}(r, f)$ *as* $N(r, f)$ *with* $n(r, f)$ *substituted for* $\overline{n}(r, f)$*.*

Examples 3.1.6

Since each of the poles of $\tan z$ is simple, $n(r, \tan z) = \overline{n}(r, \tan z)$. We saw earlier that $n(k\pi, \csc^2 z) = 2 + 4k$. Thus, $\overline{n}(k\pi, \csc^2 z) = 1 + 2k$, since each of the poles is double. From this, we see that

$$\overline{N}(k\pi, \csc^2 z) = \sum_{n=1}^{k} \int_{(n-1)\pi}^{n\pi} (2n-2)\frac{dt}{t} + \log(k\pi) = \sum_{n=1}^{k} (2n-2) \log \frac{n}{n-1} + \log(k\pi).$$

An equivalent for $N(\pi, \csc^2 z)$ would yield exactly double this result.

Theorem 3.1.7

$$T(r, f') \leq T(r, f) + \overline{N}(r, f) + S(r, f).$$

This follows from the following two formulae

$$N(r, f') = N(r, f) + \overline{N}(r, f), \quad m(r, f') \leq m(r, f) + m\left(r, \frac{f'}{f}\right),$$

and that $m(r, f'/f) = S(r, f)$.

3.2 Statement of the Second Fundamental Theorem

We will now state and prove the Second Fundamental Theorem. Note that we here omit the estimation of $S(r, f)$, but this can be found in [6], pages 34-36. We begin by defining a function.

Definition 3.2.1

$$\begin{aligned} N_1(r,f) &:= 2N(r,f) - N(r,f') + N\left(r, \frac{1}{f'}\right) \\ &= N(r,f) - \overline{N}(r,f) + N\left(r, \frac{1}{f'}\right) \end{aligned}$$

The function f is one-to-one on some neighbourhood of z_0 if and only if either $f(z_0)$ is finite and $f'(z_0) \neq 0$, or f has a simple pole at z_0. The function $N_1(r,f)$ thus counts the multiple points of f in the following way: suppose f has an a-point of multiplicity p at $z = z_0$. Then, by Rouché's Theorem, all values sufficiently close to a are taken p times near z_0. Thus, z_0 is a multiple a-point of order $p-1$, and contributes $p-1$ to $n_1(r,f)$. Note that this function is non-negative.

We will use this definition in Sections 3 and 5. Section 4 uses a different definition of $N_1(r,f)$, as given by Hayman, whose notation we retain.

Theorem 3.2.2 - The Second Fundamental Theorem

Let f be meromorphic in the plane. Then given any k distinct values b_j in $\mathbb{C}^*$, we have that

$$\sum_{j=1}^{k} m(r, b_j, f) \leq 2T(r,f) - N_1(r,f) + S(r,f), \tag{3.3}$$

or, equivalently,

$$(k-2)T(r,f) \leq \sum_{j=1}^{k} \overline{N}(r, b_j, f) + S(r,f), \tag{3.4}$$

Proof:

Let $a_1, \ldots, a_k$ be k distinct finite points in the plane, and let

$$P(f) := \prod_{j=1}^{k} (f - a_j).$$

be a polynomial in f of degree k. Then, using partial fractions,

$$\frac{1}{P(f)} = \sum_{j=1}^{k} \frac{c_j}{f - a_j}$$

for some constants c_j. Then, by the properties of $\log^+$ and the Lemma of the Logarithmic Derivative (3.2),

$$m\left(r, \frac{f'}{P(f)}\right) = m\left(r, \sum_{j=1}^{k} \frac{c_j f'}{f - a_j}\right) \leq \sum_{j=1}^{k} m\left(r, \frac{f'}{f - a_j}\right) + O(1) = S(r, f),$$

and so

$$\begin{aligned} m\left(r, \frac{1}{P(f)}\right) = m\left(r, \frac{f'}{P(f)} \frac{1}{f'}\right) &\leq m(r, 0, f') + m\left(r, \frac{f'}{P(f)}\right) + O(1) \\ &= m(r, 0, f') + S(r, f). \end{aligned}$$

Clearly,

$$N\left(r, \frac{1}{P(f)}\right) = N(r, 0, P(f)) = \sum_{j=1}^{k} N(r, a_j, f).$$

Now, it is obvious that $P(f)$ has a pole only if f has a pole, and since $P(f)$ has degree k, it is clear that $N(r, P(f)) = N(r, f^k)$. On the other hand, as $|f| \to \infty$, $P(f) \sim f^k$, and so $m(r, P(f)) = m(r, f^k) + O(1)$. By our earlier examples, $T(r, f^k) = kT(r, f)$, and so $T(r, P(f)) = kT(r, f) + S(r, f)$. Now, using the First Fundamental Theorem (2.20),

$$\begin{aligned} T(r, f') &= m(r, 0, f') + N(r, 0, f') + O(1) \\ &\geq m(r, 0, P(f)) + N(r, 0, f') + S(r, f) \\ &= T(r, P(f)) - N(r, 0, P(f)) + N(r, 0, f') + S(r, f) \\ &= kT(r, f) - \sum_{j=1}^{k} N(r, a_j, f) + N(r, 0, f') + S(r, f) \\ &= \sum_{j=1}^{k} T(r, a_j, f) - \sum_{j=1}^{k} N(r, a_j, f) + N(r, 0, f') + S(r, f) \\ &= \sum_{j=1}^{k} m(r, a_j, f) + N(r, 0, f') + S(r, f). \end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{j=1}^{k} m(r, a_j, f) &\leq T(r, f') - N(r, 0, f') + S(r, f) \\
&= m(r, f') + N(r, f') - N(r, 0, f') + S(r, f) \\
&\leq m(r, f) + m\left(r, \frac{f'}{f}\right) + N(r, f') - N(r, 0, f') + S(r, f) \\
&= T(r, f) - N(r, f) + N(r, f') - N(r, 0, f') + S(r, f) \\
&= T(r, f) + N(r, f) - N_1(r, f) + S(r, f)
\end{aligned}$$

and so, adding $m(r, f)$,

$$m(r, f) + \sum_{j=1}^{k} m(r, a_j, f) \leq 2T(r, f) - N_1(r, f) + S(r, f)$$

from which (3.3) follows since $m(r, f) \equiv m(r, \infty, f)$, and we allow at most one b_j to equal infinity. To show (3.4), we add terms in $N(r, b_j, f)$ to each side, and thus, using the First Fundamental Theorem (2.20),

$$(k-2)T(r, f) \leq \sum_{j=1}^{k} N(r, b_j, f) - N_1(r, f) + S(r, f). \tag{3.5}$$

We noted earlier that if f has an a-point of multiplicity p, then this contributes $p-1$ to $n_1(r, f)$. That same point contributes p to $n(r, a, f)$, and thus we can rearrange the above equation into (3.4).

QED

3.3 The Defect Relation and Picard's Theorem

In this section, we will look at Picard's Theorem, an important theorem of the analysis of entire functions. We will show how we can prove this using the Second Fundamental Theorem, and in fact use this to generalise Picard's Theorem to meromorphic functions. We begin with a statement of the classical result.

Theorem 3.3.1 - Picard's Theorem

Let f be a non-constant transcendental entire function. Then f takes every value in $\mathbb{C}$ infinitely often, with at most one exceptional value which is taken finitely often. For example, e^z omits the value 0, but takes every other value infinitely often.

The proof of this remarkable result is omitted here, but can be found in most textbooks on Complex Analysis. There are many methods of proving this - for instance, it is possible to do so using Schottky's Theorem - but we will in this section investigate how to do so using Nevanlinna Theory. We start with a definition:

Definition 3.3.2

Let $a \in \mathbb{C}*$. We define the deficiency*, δ, as*

$$\delta(a,f) = \lim_{r\to\infty} \inf \frac{m(r,a,f)}{T(r,f)} = 1 - \lim_{r\to\infty} \sup \frac{N(r,a,f)}{T(r,f)}. \tag{3.6}$$

This deficiency is a measure of how rarely the function f takes the value a, as can be seen by the version involving $N(r,a,f)$. It is also obvious by the First Fundamental Theorem (2.20), that $\delta \in [0,1]$ since $T(r,f) = T(r,a,f) + O(1)$.

Theorem 3.3.3 - The Defect Relation

$$\sum_{a\in\mathbb{C}*} \delta(a,f) \leq 2 \tag{3.7}$$

This follows from (3.3), dividing through by $T(r,f)$, noting that $N_1(r,f)$ is non-negative and that $S(r,f) = o(T(r,f))$, before taking the $\liminf$ as $r \to \infty$. We can see that if the value a is taken only finitely often, then $\delta(a,f) = 1$. This implies Picard's Theorem, as for entire functions $\delta(\infty,f) = 1$, and so only one other value can be taken finitely often. The result itself is quite startling in its simplicity.

Proposition 3.3.4 - An alternate proof of Picard's Theorem

It is actually possible to prove Picard's Theorem just from the Second Fundamental Theorem (3.4). Suppose that f is transcendental entire and takes two distinct finite values b_1 and b_2 only finitely often. We let $k = 3$ and use $b_3 = \infty$ as f is entire. Then,

$$\begin{aligned} T(r, f) &\leq \sum_{j=1}^{3} \overline{N}(r, b_j, f) + S(r, f) \\ &= O(\log r) + S(r, f). \end{aligned}$$

However, by our assumption, f is transcendental, and so

$$O(\log r) = S(r, f) = o(T(r, f)),$$

which is a contradiction. Thus f cannot take more than one finite value only finitely often.

Theorem 3.3.5 - A generalisation of Picard's Theorem

Let f be a complex-valued transcendental function on the complex plane. Then f takes every value in $\mathbb{C}^*$ infinitely often, with at most two exceptions which may be taken finitely. For instance, $\tan z$ *omits* $\pm i$.

This result follows by the argument of the previous proposition, but without requiring $b_3 = \infty$. This generalises Picard's Theorem to meromorphic functions.

3.4 A generalised version of Fermat's Last Theorem for functions

Most children of secondary school age are aware of Fermat's Last Theorem, that for any integer $n \geq 3$, there exist no non-zero integer triples (x, y, z) such that $x^n + y^n = z^n$. The theorem, attributed to French mathematician Pierre de Fermat, remained one of Mathematics' great unsolved problems for over 300 years, before being proved in the mid-1990s by Andrew Wiles. In this section, we will look at a version of the theorem for functions.

Let f and g be complex-valued functions in the plane, and n be a positive integer. We will look for solutions to the equation

$$f^n + g^n = 1. \tag{3.8}$$

Theorem 3.4.1

Let f and g be entire non-constant functions such that $f^n + g^n = 1$ for all $z \in \mathbb{C}$, and let n be a non-negative integer. Then $n \leq 2$.

Proof:

We know that such functions exist for $n \leq 2$ - consider $\sin^2 z + \cos^2 z = 1$. So, suppose that there exist f and g such that the above relation holds for some $n \geq 3$, then $f^n - 1 = -g^n$. Suppose g has a zero of multiplicity k at some point, then g^n has a zero of multiplicity nk at that point, as does $f^n - 1$. But, we can rewrite this as

$$f^n - 1 = \prod_{j=1}^{n} \left(f - e^{2\pi i j/n}\right),$$

and thus f has an $e^{2\pi i j/n}$-point of multiplicity nk since these points are the zeros of g. We now apply the Second Fundamental Theorem (3.4) using

$$\{b_j\} = \{\infty,\ e^{2\pi iq/n} : 0 \leq q < n\},$$

noting that $S(r,f) = o(T(r,f))$, and that $\overline{N}(r,f) \equiv 0$ since f is entire.

$$\begin{aligned}
(n-1)T(r,f) &\leq \sum_{j=1}^{n} \overline{N}\left(r, e^{2\pi ij/n}, f\right) + \overline{N}(r,f) + S(r,f) \\
&\leq \frac{1}{n}\sum_{j=1}^{n} N\left(r, e^{2\pi ij/n}, f\right) + S(r,f) \\
&\leq \frac{n}{n}T(r,f) + S(r,f) \\
&= T(r,f) + S(r,f)
\end{aligned}$$

This is clearly a contradiction for $n > 2$, and thus no such f and g can exist.

Theorem 3.4.2

Let f and g be non-constant meromorphic functions such that $f^n + g^n = 1$ for all $z \in \mathbb{C}$, and let n be a non-negative integer. Then $n \leq 3$.

Proof:

Suppose the relation holds for some $n \geq 4$, then we rewrite $f^n - 1$ as before, but do not set $b_{n+1} = \infty$ as our functions can have poles here. Then,

$$\begin{aligned}
(n-2)T(r,f) &\leq \sum_{j=1}^{n} \overline{N}\left(r, e^{2\pi ij/n}, f\right) + S(r,f) \\
&\leq T(r,f) + S(r,f),
\end{aligned}$$

which yields a contradiction for $n > 3$, and thus no such f and g can exist.

We will spend the rest of this section verifying a result by Fred Gross in [5], where he gives an example of meromorphic functions f and g satisfying $f^3 + g^3 = 1$. We first however need to look into some functions he used.

We are all familiar with how the trigonometric functions sin and cos are periodic on the real axis with period 2π. We now look at functions in the complex plane which are periodic in two dimensions - the elliptic functions. We define the two periods of the function as ω_1 and ω_2. It is clear that if the two period dimensions do not have the same argument, then the function "tiles" the complex plane with a series of parallelograms. We use results from [9] as the start of our investigations.

Proposition 3.4.3

There exists a doubly-periodic function $\wp(z)$ *with primitive periods* ω_1 *and* ω_2*, with a double pole at each of the points* $\omega = m\omega_1 + n\omega_2$*, for* $m, n \in \mathbb{Z}$*, and which has the expansion*

$$\wp(z) = \frac{1}{(z-\omega)^2} + a_1(z-\omega) + \dots \tag{3.9}$$

at each such pole, then the function $\wp(z)$ *is uniquely determined and has the expansion*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \tag{3.10}$$

at every point $z \neq \omega$*. This function is called the Weierstrass* $\wp$*-function.*

Proposition 3.4.4

$\wp(z)$ *solves the following differential equation:*

$$[\wp'(z)]^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \tag{3.11}$$

Here, $e_j = \wp(\omega_j/2)$ *and* $\omega_3 = \omega_1 + \omega_2$*.*

For a proof of this, see [1]. Theorem 6.4 in that same text tells us that

$e_1 + e_2 + e_3 = 0$, and thus we can rewrite the above equation as

$$[\wp'(z)]^2 = 4[\wp(z)]^3 - g_2\wp(z) - g_3, \tag{3.12}$$

where $-g_2 = 4e_1e_2 + 4e_1e_3 + 4e_2e_3$ and $g_3 = 4e_1e_2e_3$.

For the final proposition, we turn to [12].

Proposition 3.4.5

Given two constants g_2 and g_3 such that $g_2^3 \neq 27g_3^2$, it is possible to construct a Weierstrass $\wp$-function using these invariants.

We are now in a position to verify Gross's result.

Theorem 3.4.6

Let $\wp$ be the Weierstrass $\wp$-function with $g_2 = 0$ and $g_1 = 1$, then

$$f = \frac{1 + \frac{\wp'}{\sqrt{3}}}{2\wp}, \qquad g = \frac{1 - \frac{\wp'}{\sqrt{3}}}{2\wp}$$

satisfy the equation $f^3 + g^3 = 1$.

Proof:

We first expand the cubes, giving

$$\begin{aligned} f^3 &= \frac{1}{8\wp^3}\left[1 + 3\left(\frac{\wp'}{\sqrt{3}}\right) + 3\left(\frac{\wp'}{\sqrt{3}}\right)^2 + \left(\frac{\wp'}{\sqrt{3}}\right)^3\right] \\ &= \frac{1}{8\wp^3}\left[1 + \sqrt{3}\wp' + (\wp')^2 - \left(\frac{\wp'}{\sqrt{3}}\right)^3\right], \end{aligned}$$

and

$$\begin{aligned} g^3 &= \frac{1}{8\wp^3}\left[1 - 3\left(\frac{\wp'}{\sqrt{3}}\right) + 3\left(\frac{\wp'}{\sqrt{3}}\right)^2 - \left(\frac{\wp'}{\sqrt{3}}\right)^3\right] \\ &= \frac{1}{8\wp^3}\left[1 - \sqrt{3}\wp' + (\wp')^2 - \left(\frac{\wp'}{\sqrt{3}}\right)^3\right]. \end{aligned}$$

Adding these yields

$$\begin{aligned} f^3 + g^3 &= \frac{2}{8\wp^3}\left[1 + (\wp')^2\right] \\ &= \frac{1}{4\wp^3}\left[1 + 4\wp^3 - g_2\wp - g_3\right]. \end{aligned}$$

Obviously, for equality, we require that the bracketed term equal $4\wp^3$. Using $g_2 = 0$ and $g_3 = 1$ as stated gives us the required equality. That they are non-constant can be shown by looking at f', although we omit this here.

QED

4 Hayman's Alternative

In this section, we will look at results by the mathematician Milloux, as depicted in [6], and Hayman's work based on this, the culmination of which is Corollary 4.2.6, which became known as *Hayman's Alternative.* It states that either the transcendental function f takes every finite value infinitely often, or every derivative takes every finite non-zero value infinitely often. This is significant because Picard's Theorem requires that three values be checked before an equation can be said to be constant, while Hayman's Alternative requires only two. We will then look at a generalisation of Hayman's results by J. K. Langley and Walter Bergweiler in their 2003 paper "Multiplicites in Hayman's Alternative" [4].

Throughout this section we will use the convention that $S(r, f) = o(T(r, f))$, and that the functions a_s are meromorphic in the plane and satisfy $T(r, a_s) = S(r, f)$. We will call these *small* functions. We begin by looking at Milloux's results.

4.1 Milloux Theory

Milloux's results provided the springboard for Hayman to prove his Alternative, as well as being the starting point for Bergweiler and Langley's generalisation. We will prove a generalised version of the Lemma of the Logarithmic Derivative (3.1.2), then prove a version of the Alternative for functions with only finite numbers of zeros and poles.

Theorem 4.1.1

Let t be a positive integer, f be a function which is meromorphic and non-constant in the plane, and let

$$\psi(z) = \sum_{s=0}^{t} a_s(z) f^{(s)}(z). \tag{4.1}$$

Then the following two equations hold:

$$m\left(r, \frac{\psi(z)}{f(z)}\right) = S(r, f) \tag{4.2}$$

$$T(r, \psi) \leq (t+1)T(r, f) + S(r, f). \tag{4.3}$$

Proof:

We will first consider the case $\psi = f^{(t)}$, and continue by induction on t. Let us assume that we have proved that

$$m\left(r, \frac{f^{(t)}}{f}\right) = S(r, f)$$

for some non-negative integer t - in fact, by the Lemma of the Logarithmic Derivative, we know this is true for $t = 1$. Then, using the properties of $\log^+$, we can see that

$$m\left(r, f^{(t)}\right) \leq m\left(r, \frac{f^{(t)}}{f}\right) + m(r, f) = m(r, f) + S(r, f).$$

Now, suppose f has a pole of order $p \geq 1$ at some point. Then, at that point, $f^{(t)}$ has a pole of order $t + p \leq (t+1)p$. Thus,

$$N\left(r, f^t\right) \leq (t+1)N(r, f).$$

Adding the two equations, we find

$$\begin{aligned} T\left(r, f^{(t)}\right) = m\left(r, f^{(t)}\right) + N\left(r, f^{(t)}\right) &\leq m(r, f) + S(r, f) + (t+1)N(r, f) \\ &\leq (t+1)T(r, f) + S(r, f) \end{aligned}$$

which proves (4.3) in this specific case. Now, note that

$$m\left(r, \frac{f^{(t+1)}}{f^{(t)}}\right) = S(r, f^{(t)}) = o(T(r, f^{(t)})) = o(T(r, f)) = S(r, f),$$

and thus,

$$m\left(r, \frac{f^{(t+1)}}{f}\right) \leq m\left(r, \frac{f^{(t+1)}}{f^{(t)}}\right) + m\left(r, \frac{f^{(t)}}{f}\right) = S(r,f) + S(r,f) = S(r,f),$$

which completes the proof for the case $\psi = f^{(k)}$.

Now, let us return to our general definition of ψ. Using Lemma 2.3.4,

$$\begin{aligned} m\left(r, \frac{\psi}{f}\right) &\leq \sum_{s=0}^{t} m\left(r, \frac{a_j f^{(s)}}{f}\right) + t\log 2 \\ &\leq \sum_{s=0}^{t} \left[m(r, a_s) + m\left(r, \frac{f^{(s)}}{f}\right)\right] + O(1) \\ &= \sum_{s=0}^{t} S(r,f) + O(1) = S(r,f), \end{aligned}$$

which completes the proof of (4.2). Now, note again that

$$m(r,\psi) \leq m(r,f) + m\left(r, \frac{\psi}{f}\right) = m(r,f) + S(r,f).$$

Further, if f has a pole of order $p \geq 1$ at some point, and the a_s have poles of order at most n at that point, then ψ has a pole of order at most $p+t+n \leq (t+1)p+n$ at that point, and thus

$$\begin{aligned} N(r,\psi) &\leq (t+1)N(r,f) + \sum_{s=0}^{t} N(r, a_s) \\ &\leq (t+1)N(r,f) + S(r,f), \end{aligned}$$

which combined with the previous equation, yields (4.3).

QED

Theorem 4.1.2

Let f be meromorphic and non-constant in the plane, and ψ as defined by (4.1) also be non-constant. Then,

$$T(r,f) < \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right) + S(r,f), \quad (4.4)$$

where $N_0(r, 1/\psi')$ *counts only zeros of* ψ' *which are not multiple 1-points of* ψ.

Proof:

We first apply the Second Fundamental Theorem (3.3) to ψ, using $k = 3$, and $b_j \in \{0,\ 1,\ \infty\}$, giving

$$m(r,\psi) + m\left(r, \frac{1}{\psi}\right) + m\left(r, \frac{1}{\psi - 1}\right) \leq 2T(r,\psi) - N_1(r,\psi) + S(r,\psi). \qquad (4.5)$$

Note that since $T(r,\psi) = O(T(r,f))$ by (4.3), it follows that $S(r,\psi) = S(r,f)$. Using the First Fundamental Theorem (2.4.2) and Definition 2.3.1, we see that

$$2T(r,\psi) - N_1(r,\psi) = m(r,\psi) + N(r,\psi) + m\left(r, \frac{1}{\psi - 1}\right) - N\left(1, \frac{1}{\psi'}\right) +$$

$$+N\left(r, \frac{1}{\psi - 1}\right) - 2N(r,\psi) + N(r,\psi') + O(1). \qquad (4.6)$$

If ψ has a pole of order p at some point, then ψ' has a pole of order $p+1$ at that point, and since such poles can occur only at poles of f or a_s,

$$N(r,\psi') - N(r,\psi) = \overline{N}(r,\psi) \leq \overline{N}(r,f) + \sum_{s=0}^{t} \overline{N}(r,a_s) = \overline{N}(r,f) + S(r,f).$$

Now, if ψ has a 1-point of order p, then ψ' has a zero of order $p-1$, and so

$$N\left(r, \frac{1}{\psi - 1}\right) - N\left(r, \frac{1}{\psi'}\right) = \overline{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right),$$

where the first term on the right takes account of the multiple 1-points of ψ, each counted once, and the second takes account of the zeroes of ψ' which do not correspond to 1-points of ψ. Putting this all together, and substituting back into (4.5), we see that

$$m\left(r, \frac{1}{\psi}\right) \leq \overline{N}(r,f) - \overline{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right) + S(r,f). \qquad (4.7)$$

Now, using the First Fundamental Theorem again, and applying (4.2),

$$\begin{aligned} T(r,f) &= N\left(r,\frac{1}{f}\right) + m\left(r,\frac{1}{f}\right) + O(1) \\ &= N\left(r,\frac{1}{f}\right) + m\left(r,\frac{1}{\psi}\right) + m\left(r,\frac{\psi}{f}\right) + O(1) \\ &= N\left(r,\frac{1}{f}\right) + m\left(r,\frac{1}{\psi}\right) + S(r,f), \end{aligned}$$

from which the theorem follows when (4.7) is substituted in.

QED

We can then use this theorem to prove a quite interesting result.

Corollary 4.1.3

Let f be meromorphic transcendental and non-constant in the plane, with only a finite number of zeros and poles. Then every function ψ, as defined in (4.1), assumes every finite complex value, except possibly zero, infinitely often, or else is identically constant.

Proof:

We can have a case where ψ is identially constant - for instance, let $\psi = f' - f$, and $f = e^z + c$ for some constant c. Now, suppose that ψ is not constant. Then (4.4), combined with the fact that if f has finitely many poles then $N(r,f) = O(\log r)$, shows us that

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r,\frac{1}{\psi-1}\right) - N_0\left(r,\frac{1}{\psi'}\right) + S(r,f) + O(\log r) \\ &\leq \overline{N}\left(r,\frac{1}{\psi-1}\right) + S(r,f) + O(\log r), \end{aligned}$$

or, equivalently,

$$(1+o(1))T(r,f) \leq \overline{N}\left(r,\frac{1}{\psi-1}\right) + O(\log r) \tag{4.8}$$

as $r \to \infty$. This shows that ψ must have infinitely many 1-points, since if it were not true, then we would have $T(r, f) = O(\log r)$, which would then imply, by Lemma 2.3.9, that f was a rational function, which contradicts our assumption that f is transcendental.

We can replace now ψ by $\varphi = \psi/\omega$ for some finite, non-zero ω, and thus find that φ has infinitely many 1-points, or equivalently that ψ has infinitely many ω-points, and thus that ψ takes every finite value non-zero infinitely often.

QED

4.2 Results in a special case

In this section, we will focus on the special case $\psi = f^{(t)}$, and prove several results, which we will then try to generalise in the next section. We will return, in a manner, to the Defect Relation, Theorem 3.3.3. We will define a new function, and hence prove that the sum of this function over all values in $\mathbb{C}$ is bounded. We will then prove a stronger version of this theorem, give a new bound for $T(r, f)$ in our special case, and hence prove Hayman's Alternative.

Definition 4.2.1

We define the following variation upon the deficiency:

$$\Theta(a, f) = 1 - \lim_{r\to\infty} \sup \frac{\overline{N}(r, a, f)}{T(r, f)}.$$

It is clear that $\Theta(a, f) \geq \delta(a, f)$, and also that $\Theta(a, f) = 1$ if f takes the value a only finitely often.

We will now use this definition to prove a version of the Defect Relation.

Theorem 4.2.2

Let f be meromorphic in the plane, then

$$\sum_{a\in\mathbb{C}^*} \Theta(a,f) \leq 2. \tag{4.9}$$

Proof:

We will use a slightly modified version of the Second Fundamental Theorem. Let $b_j \in \mathbb{C}$ for all j, then

$$(k-1)T(r,f) \leq \sum_{j=1}^{k} \overline{N}(r,b_j,f) + \overline{N}(r,\infty,f) + S(r,f).$$

We divide through by $T(r,f)$, remembering that $S(r,f) = o(T(r,f))$, and find, taking the $\limsup$ as $r \to \infty$,

$$\begin{aligned} k-1 &\leq \sum_{j=1}^{k} \frac{\overline{N}(r,b_j,f)}{T(r,f)} + \frac{\overline{N}(r,\infty,f)}{T(r,f)} \\ &\leq \sum_{j=1}^{k} [1-\Theta(b_j,f)] + [1-\Theta(\infty,f)] \end{aligned}$$

which in turn yields

$$\sum_{j=1}^{k} \Theta(b_j,f) + \Theta(\infty,f) \leq 2, \tag{4.10}$$

from which (4.9) follows.

QED

It is clear that this is a slightly stronger version of the defect relation. We now prove a yet stronger version in a special case.

Theorem 4.2.3

Let f be a transcendental meromorphic function in the plane. Then

$$\sum_{a\in\mathbb{C}} \Theta\left(a, f^{(t)}\right) \leq 1 + \frac{1}{t+1}, \tag{4.11}$$

and so $f^{(t)}$ assumes every finite value infinitely often, with at most one exception.

Proof:

Suppose f has a pole of order $p \geq 1$ at some point, then $f^{(t)}$ has a pole of order $p+t \geq t+1$ at that point. Hence,

$$\overline{N}\left(r, f^{(t)}\right) \leq \frac{1}{t+1} N\left(r, f^{(t)}\right) \leq \frac{1}{t+1} T\left(r, f^{(t)}\right),$$

and thus,

$$\Theta\left(\infty, f^{(t)}\right) \geq \frac{t}{t+1}.$$

We now substitute this into (4.10), from which the result follows.

QED

When we combine this result with Corollary 4.1.3, it is clear that if f has only finitely many zeros and poles, then any value taken only finitely often by ψ must be 0. We will now proceed to show that this is true regardless of the number of poles of f. First though, we need some lemmas.

Lemma 4.2.4

Let f be transcendental and meromorphic in the plane, $\psi = f^{(t)}$, and $N_0(r, 1/\psi')$ be defined as in Theorem 4.1.2. Then

$$tN_1(r,f) \leq \overline{N}_2(r,f) + \overline{N}\left(r, \frac{1}{\psi-1}\right) + N_0\left(r, \frac{1}{\psi'}\right) + S(r,f), \tag{4.12}$$

where $N_1(r,f)$ counts the simple poles of f, and $\overline{N}_2(r,f)$ counts the number of multiple poles of f, not including multiplicity.

Proof:

We first define the function

$$g = \frac{\left(f^{(t+1)}\right)^{t+1}}{\left(1-f^{(t)}\right)^{t+2}} = \frac{(\psi')^{t+1}}{(1-\psi)^{t+2}}.$$

Suppose f has a simple pole at z_0, i.e. $f(z_0) = a(z-z_0)^{-1} + O(1)$ for some $a \neq 0$. Then, differentiating t times,

$$f^{(t)}(z_0) = \frac{(-1)^t a t!}{(z-z_0)^{t+1}} \left(1 + O\left((z-z_0)^{t+1}\right)\right).$$

Differentiating again, and then susbtituting into g, we find that

$$g = \frac{(-1)^{t+1}(t+1)^{t+1}}{at!} \left(1 + O\left((z-z_0)^{t+1}\right)\right).$$

Thus, at a simple pole of f, $g \neq 0, \infty$, but g' has a zero of order at least t. Now we apply (2.18) to g'/g, assuming g to be non-constant, giving

$$\begin{aligned} m\left(r, \frac{g'}{g}\right) - m\left(r, \frac{g}{g'}\right) + O(1) &= N\left(r, \frac{g}{g'}\right) - N\left(r, \frac{g'}{g}\right) \\ &= N(r,g) + N\left(r, \frac{1}{g'}\right) - N(r,g') - N\left(r, \frac{1}{g}\right) \\ &= N\left(r, \frac{1}{g'}\right) - N\left(r, \frac{1}{g}\right) - \overline{N}(r,g) \\ &= N_0\left(r, \frac{1}{g'}\right) - \overline{N}\left(r, \frac{1}{g}\right) - \overline{N}(r,g) \end{aligned}$$

where $N_0(r, 1/g')$ counts only zeros of g' which are not also zeroes of g.

Thus, using the above, and the property that $m(r, g/g')$ is non-negative

$$tN_1(r,f) \leq N_0\left(r, \frac{1}{g'}\right) \leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r,g) + m\left(r, \frac{g'}{g}\right) + O(1).$$

Note that by the Lemma of the Logarithmic Derivative, the last two terms in the above equation are $S(r,f)$. Now, zeros and poles of g can only occur at multiple poles of f, 1-points of ψ, or zeros of ψ' which are not 1-points of ψ, and so

$$\overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r,g) \leq \overline{N}\left(r, \frac{1}{\psi - 1}\right) + \overline{N}_2(r,f) + N_0\left(r, \frac{1}{\psi'}\right),$$

which proves the lemma.

QED

We are now in a position to prove the main result and corollary of this section.

Theorem 4.2.5

Let f be transcendental and meromorphic in the plane. Then,

$$T(r,f) \leq \left(2+\frac{1}{t}\right) N\left(r,\frac{1}{f}\right) + \left(2+\frac{2}{t}\right) \overline{N}\left(r,\frac{1}{f^{(t)}-1}\right) + S(r,f) \qquad (4.13)$$

as $r \to \infty$.

Proof:

We start by noting that in $N(r,f)$, multiple poles are counted at least twice, and then apply (4.4) using $\psi = f^{(t)}$.

$$\begin{aligned} N_1(r,f) &+ 2\overline{N}_2(r,f) \leq T(r,f) \\ &\leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{\psi-1}\right) - N_0\left(r,\frac{1}{\psi'}\right) + S(r,f) \end{aligned}$$

Since $\overline{N}(r,f) = N_1(r,f) + \overline{N}_2(r,f)$, we can further say that

$$\overline{N}_2(r,f) \leq N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{\psi-1}\right) - N_0\left(r,\frac{1}{\psi'}\right) + S(r,f).$$

We now substitute this into (4.12), giving

$$tN_1(r,f) \leq N\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{\psi-1}\right) + S(r,f).$$

Finally, we combine this with the previous equation to give

$$\begin{aligned} \overline{N}(r,f) &= N_1(r,f) + \overline{N}_2(r,f) \\ &\leq \left(2+\frac{1}{t}\right) N\left(r,\frac{1}{f}\right) + \left(1+\frac{2}{t}\right) \overline{N}\left(r,\frac{1}{\psi-1}\right) + S(r,f). \end{aligned}$$

We then substitute this back into (4.4), and since $N_0(r,1/\psi') \geq 0$, the result follows.

QED

Corollary 4.2.6

Let f be transcendental and meromorphic in the plane. Then either f assumes every finite value infinitely often, or $f^{(t)}$ assumes every finite value except possibly zero infinitely often. This result is known as Hayman's Alternative.

Proof:

Let ω_1 and ω_2 be complex numbers with $\omega_2 \neq 0$, and define

$$F = \frac{f - \omega_1}{\omega_2}. \tag{4.14}$$

Then $T(r, F) = T(r, f) + O(1)$, and so, applying (4.13),

$$\begin{aligned} T(r,f) &\leq \left(2 + \frac{1}{t}\right) N\left(r, \frac{1}{F}\right) + \left(2 + \frac{2}{t}\right) \overline{N}\left(r, \frac{1}{F^{(t)} - 1}\right) + S(r,f) \\ &= \left(2 + \frac{1}{t}\right) N\left(r, \frac{1}{f - \omega_1}\right) + \left(2 + \frac{2}{t}\right) \overline{N}\left(r, \frac{1}{f^{(t)} - \omega_2}\right) + S(r,f) \end{aligned}$$

If $f = \omega_1$ and $f^{(t)} = \omega_2$ have only finitely many solutions, then $T(r, f) = O(\log r)$, which implies that f is rational, which contradicts the hypothesis that f is transcendental.

QED

We can refine this result in the following way, as noted by Pang, Nevo and Zalcman in [10].

Corollary 4.2.7

Let f be a meromorphic function in the plane. If $f \neq 0$ and $f^{(t)} \neq 1$ for some fixed positive integer t, then f is constant.

We note that we can choose the values 0 and 1 by using (4.14).

Examples 4.2.8

(i) Let $f = e^z$. Since $f \neq 0$, all derivatives of f, which are in fact equal to f, take every finite non-zero value infinitely often.

(ii) The trigonometric functions $\sin z$ and $\cos z$ take every finite value infinitely often. This is clear since here $f^{(4)} = f$, and we know that f takes the value zero infinitely often.

(iii) Since we know that $\tan z \neq \pm i$, we now know that all derivatives of $\tan z$ take all finite non-zero values infinitely often. The same is true for derivatives of the functions $\csc z$, and $\sec z$, neither of which take the value zero in the plane, and $\cot z$, which also fails to take the values $\pm i$.

4.3 Generalisation of the results

We will now look at a paper by Walter Bergweiler and J. K. Langley [4] entitled *Multiplicities in Hayman's Alternative*, where the authors use the results from the previous section in a re-generalisation of ψ to a form only slightly different from that of (4.1), in that we require a_t be identically constant. We will start with a statement of the result, then various lemmas which will allow us to prove the result.

Theorem 4.3.1

Let f be meromorphic and non-constant in the plane. Let k be a positive integer, and let the linear differential operator L_t be defined as

$$L_t = \frac{d^t}{dz^t} + \sum_{s=0}^{t-1} a_s \frac{d^s}{dz^s}$$

where a_s *are small functions, and* $\psi = L_t(f)$ *is assumed to be non-constant. If* $a_s \not\equiv 0$ *for some* $0 \leq s \leq t-1$*, then let*

$$Q = \max\{s : 0 \leq s \leq t-1,\ \ a_s \not\equiv 0\}, \qquad q = \max\{1,\ t-1-Q\},$$

otherwise let $Q = -1$ *and* $q = t$*. Then one of the following two conditions holds.*

(i) *The following inequality holds:*

$$T(r,f) \leq 2N^{(t+1)}\left(r,\frac{1}{f}\right) + \frac{1}{q}N^{(t+2)}\left(r,\frac{1}{f}\right) + \left(2+\frac{2}{q}\right)\overline{N}\left(r,\frac{1}{\psi-1}\right) + S(r,f), \tag{4.15}$$

where $N^{(p)}(r,1/f)$*,* $p \in \mathbb{N}$*, counts the zeros of* f*, but with zeros of order* p *or greater counted only* p *times.*

(ii) *There exists a function* H*, meromorphic in the plane, satisfying the following equation*

$$\beta(z) = \frac{H''(z)}{H'(z)} = \frac{2a_{t-1}(z)}{t(t+1)}, \tag{4.16}$$

such that the following equations hold:

$$f = \frac{(H-\omega)^{t+1}}{t!H(H')^t} \tag{4.17}$$

$$L_t = \left(\frac{d}{dz}+\beta\right)\ldots\left(\frac{d}{dz}+t\beta\right) \tag{4.18}$$

$$\psi = L_t(f) = 1 - \frac{1}{H^{t+1}} \tag{4.19}$$

in which ω *is a* $(t+1)^{th}$ *root of unity.*

Note that if $a_{t-1} \equiv 0$*, then* (i) *holds. We can remove the restriction that the coefficient of* $f^{(t)}$ *be 1 by setting* $g = a_t f$*, which then gives* $g^{(t)} = a_t f^{(t)}$*, and changing the other* a_s *accordingly.*

We will prove this theorem in three stages, with lemmas before each stage.

Examples 4.3.2

Let us look at some examples of H as defined in Theorem 4.3.1. b and c are assumed to be finite constants.

(i) First, let $\beta = 0$, which implies that $H'' = 0$, and so $H = bz + c$.

(ii) We now consider $\beta = k$, where k is a finite non-zero constant. Then $H'' = kH'$, and so $H = be^{kz}$.

(iii) Finally let us consider $\beta = 2z$. Then $H'' = 2zH'$, and so $H' = be^{z^2}$, the integration of which is omitted here.

Lemma 4.3.3

Let f and ψ be as in Theorem 4.3.1. Then

$$N\left(r, \frac{1}{f}\right) + N^*\left(r, \frac{1}{\psi'}\right) \leq N^{(t+1)}\left(r, \frac{1}{f}\right) + N_0\left(r, \frac{1}{\psi'}\right) + S(r, f) \tag{4.20}$$

where $N_0(r, 1/\psi')$ counts zeros of ψ' which are not 1-points of ψ, and $N^*(r, 1/\psi')$ counts zeros of ψ' which are neither 1-points of ψ, nor zeros of f of order $\geq t+2$.

Proof:

Suppose first that at some point z_0, at least one of ψ' and f has a zero. Now suppose that f has a zero of multiplicity m at z_0, where $m = 0$ if $f(z_0) \neq 0$, and further suppose that $m \leq t + 1$. Then z_0 contributes the same to $n_0(r, 1/\psi')$ as to $n^*(r, 1/\psi')$, and it contributes m to each of $n(r, 1/f)$ and $n^{(t+1)}(r, 1/f)$, and so the inequality holds in this case.

Now let $m \geq t+2$. It can be shown that $N(r, 1/f) \leq N(r, \psi/f) + N(r, 1/\psi)$. We let $p_s \geq 0$ be the order of the pole of $a_s(z_0)$, and let μ be the order of the zero of $\psi(z_0)$. Since $f^{(s)}/f$ has poles of order at most s, and

$$\frac{\psi}{f} = \frac{f^{(t)}}{f} + \sum_{s=0}^{t-1} a_s \frac{f^{(s)}}{f},$$

we see that

$$m \leq t + \mu + \sum_{s=0}^{t-1} p_s = (t+1) + (\mu - 1) + \sum_{s=0}^{t-1} p_s.$$

Thus we see that z_0 contributes m to $n(r, 1/f) + n^*(r, 1/\psi')$, and at least $(t+1) + (\mu - 1)$ to the two n terms on the right hand side. Therefore, we get

$$N\left(r, \frac{1}{f}\right) + N^*\left(r, \frac{1}{\psi'}\right) \leq N^{(t+1)}\left(r, \frac{1}{f}\right) + N_0\left(r, \frac{1}{\psi'}\right) + \sum_{s=0}^{t-1} N(r, a_s).$$

But, by our definition, $T(r, a_s) = S(r, f)$, and so the lemma follows.

QED

We can now do the first part of the proof of Theorem 4.3.1.

Proof of Theorem 4.3.1, Part I

Let f and ψ be as defined earlier. We start with Milloux's inequality (4.4), which we proved earlier. Using (4.20), this inequality becomes

$$T(r, f) \leq \overline{N}(r, f) + N^{(t+1)}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) - N^*\left(r, \frac{1}{\psi'}\right) + S(r, f) \quad (4.21)$$

We now recall the definitions of $N_1(r, f)$ and $\overline{N}_2(r, f)$ from Lemma 4.2.4, and use the following relations:

$$\overline{N}(r, f) = N_1(r, f) + \overline{N}_2(r, f) \quad (4.22)$$

$$N_1(r, f) + 2\overline{N}_2(r, f) \leq N(r, f) \leq T(r, f).$$

We then substitute these in to (4.21) and obtain

$$\overline{N}_2(r,f) \leq N^{(t+1)}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{\psi-1}\right) - N^*\left(r,\frac{1}{\psi'}\right) + S(r,f). \tag{4.23}$$

Note that if $N_1(r,f) = S(r,f)$, then we get an equation stronger than that which we are trying to prove, and thus we can assume that $N_1(r,f) \neq S(r,f)$.

This ends the first part of the proof. Before we can proceed, we need to introduce a new function, M. We will then complete the proof of Theorem 4.3.1 in two stages - first for $M \not\equiv 0$, and then for $M \equiv 0$.

Lemma 4.3.4

Let M be defined as

$$M = (t+1)\frac{\psi''}{\psi'} - (t+2)\frac{\psi'}{\psi-1} - \frac{2a_{t-1}}{t}, \tag{4.24}$$

and let q and Q be defined as before. Let f have a simple pole at some z_0 where no a_s has a pole. Then M has a zero of multiplicity q at z_0.

Proof:

We first assume that $Q = k-1$, such that $q = 1$. Then, as $z \to z_0$, $f = \nu(z-z_0)^{-1} + O(1)$ for some non-zero constant ν, and near z_0 we can write $\psi - 1$ and ψ' as

$$\begin{aligned} \psi - 1 &= \frac{(-1)^t t!\nu}{(z-z_0)^{t+1}} + \frac{(-1)^{t-1}(t-1)!\nu a_{t-1}(z_0)}{(z-z_0)^t} + O\left(\frac{1}{(z-z_0)^{t-1}}\right) \\ \psi' &= \frac{(-1)^{t+1}(t+1)!\nu}{(z-z_0)^{t+2}} + \frac{(-1)^t t!\nu a_{t-1}(z_0)}{(z-z_0)^{t+1}} + O\left(\frac{1}{(z-z_0)^t}\right), \end{aligned}$$

from which it follows that as $z \to z_0$,

$$\begin{aligned} \frac{\psi'}{\psi-1} &= -\frac{t+1}{z-z_0} - \frac{a_{t-1}}{t} + O(z-b) \\ \frac{\psi''}{\psi'} &= -\frac{t+2}{z-z_0} - \frac{a_{t-1}}{t+1} + O(z-b), \end{aligned}$$

and thus we see that $M(z_0) = 0$, and M has a zero of order at least $q = 1$ at z_0.

Now let $0 \leq Q \leq t-1$ such that $a_{t-1} \equiv 0$ and $q = t-1-Q$. Then near z_0,

$$\begin{aligned} \psi - 1 &= \frac{(-1)^t t! \nu}{(z-z_0)^{t+1}} + O\left(\frac{1}{(z-z_0)^{Q+1}}\right) \\ \psi' &= \frac{(-1)^{t+1}(t+1)!\nu}{(z-z_0)^{t+2}} + O\left(\frac{1}{(z-z_0)^{Q+2}}\right), \end{aligned}$$

which gives, as $z \to z_0$,

$$\frac{\psi'}{\psi - 1} = -\frac{t+1}{z-z_0} + O\left((z-z_0)^q\right), \qquad \frac{\psi''}{\psi'} = -\frac{t+2}{z-z_0} + O\left((z-z_0)^q\right), \tag{4.25}$$

from which it is obvious that M has a pole of order at least q at z_0. Finally, suppose that $Q = -1$ (i.e. $\psi = f^{(t)}$), such that $q = t$, then as $z \to z_0$,

$$\begin{aligned} \psi - 1 &= \frac{(-1)^t t! \nu}{(z-z_0)^{t+1}} + O(1) \\ \psi' &= \frac{(-1)^{t+1}(t+1)!\nu}{(z-z_0)^{t+2}} + O(1), \end{aligned}$$

from which (4.25) follows.

QED

Proof of Theorem 4.3.1 - Part II

Suppose that M, as defined in (4.24), is not identically 0. By the Lemma of the Logarithmic Derivative, $m(r, M) = S(r, \psi) = S(r, f)$, and so, by Lemma 4.3.4 and the First Fundamental Theorem,

$$qN_1(r, f) \leq N\left(r, \frac{1}{M}\right) + S(r, f) \leq T(r, M) + S(r, f) \leq N(r, M) + S(r, f).$$

However, it is evident that poles of $M + 2a_{t-1}/t$ can only ever be simple, and must arise from one of the following conditions: a pole of a_s, a multiple pole of f, a 1-point of ψ, a zero of ψ' which contributes to $N^*(r, 1/\psi')$, or a zero of f of multiplicity at least $t+2$. Since a zero of f of multiplicity at least $t+2$ contributes

1 to $n^{(t+2)}(r,1/f) - n^{(t+1)}(r,1/f)$, when these are combined we have

$$\begin{aligned} qN_1(r,f) \le \overline{N}_2(r,f) &+ \overline{N}\left(r,\frac{1}{\psi-1}\right) + N^*\left(r,\frac{1}{\psi}\right) + \\ &+ N^{(t+2)}\left(r,\frac{1}{f}\right) - N^{(t+1)}\left(r,\frac{1}{f}\right) + S(r,f). \end{aligned}$$

We now substitute in (4.23), and obtain

$$qN_1(r,f) \le N^{(t+2)}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{\psi-1}\right) + S(r,f).$$

We substitute this, (4.22) and (4.23) into (4.21), from which we obtain the required result.

We will now turn our attentions to the case of $M \equiv 0$, which is unfortunately harder to prove. Let β be defined as in (4.16). As we assume that $N_1(r,f) \neq S(r,f)$, we can say that there exists a simple pole z_0 of f, and that there exists a small connected region R around that point in which all the small functions a_s are analytic. It then follows that we can define an analytic branch H of $(1-\psi)^{-1/(t+1)}$ in this region, with a simple zero at z_0. Since $M \equiv 0$, it follows that

$$(t+1)\frac{\psi''}{\psi'} - (t+2)\frac{\psi'}{\psi-1} = \frac{2a_{t-1}}{t},$$

from which we see that H is a solution to (4.16). Furthermore, we have $\psi = L_t(f) = 1 - \dfrac{1}{H^{t+1}}$ as required.

We now define two linear operators by

$$J = \left(\frac{d}{dz} + \beta\right)\dots\left(\frac{d}{dz} + t\beta\right), \qquad K = L_t - J.$$

We further define a function G, meromorphic in R, by

$$G = \frac{H^{t+1} - (-1)^t}{t!H(H')^t}.$$

Then using (4.16), it can be deduced that

$$\left(\frac{d}{dz} + t\beta\right)(G) = \frac{1}{(H')^{t-1}}\frac{d}{dH}\left(\frac{H^{t+1} - (-1)^t}{t!H}\right),$$

and by repetition,

$$J(G) = 1 - \frac{1}{H^{t+1}} = \psi.$$

This completes part two of the proof. We now need more lemmas.

Lemma 4.3.5

The operator $K = L_t - J$ is the zero operator, and thus $L_t = J$.

Proof:

Suppose that K is not the zero operator. Then, since the coefficients of K are small functions and $N_1(r, f) \neq S(r, f)$, we can choose some point z_0 such that $K(f)$ has a pole at that point. Then as $z \to z_0$,

$$f(z) = \frac{c}{z - z_0} + O(1), \qquad H(z) = \nu(z - z_0) + O\left((z - z_0)^2\right),$$

and so a comparison of the Laurent series for $J(G)$ gives

$$c = \frac{(-1)^{t+1}}{t!\nu^{t+1}}.$$

However, as $z \to z_0$,

$$G(z) = \frac{(-1)^{t+1}}{t!\nu^{t+1}(z - z_0)} = \frac{c}{z - z_0} + O(1),$$

and so $N - f$ is analytic at z_0. However,

$$K(f) = L_t(f) - J(f) = \psi - J(f) = J(N - f),$$

which gives a contradiction since $K(f)$ has a pole at z_0, yet $J(N - f)$ is regular at the same point. Thus K must be the zero operator.

QED

Lemma 4.3.6

Either (4.15) holds, or H is meromorphic in the plane.

Proof:

Let $V = H^{t+1}$, and

$$\omega = \left(\frac{d}{dz} + 2\beta\right)\dots\left(\frac{d}{dz} + t\beta\right) f$$

where $\omega = f$ if $t = 1$. Then it is clear that

$$\left(\frac{d}{dz} + \beta\right)\omega = \psi = 1 - \frac{1}{H^{t+1}} = 1 - \frac{1}{V},$$

and so V is meromorphic in the plane. Integrating on R,

$$\omega = \frac{(t+1)(tV+1)}{tV'} + \frac{c}{H'}$$

for some constant c. If $c \neq 0$, then H is meromorphic in the plane. If $c = 0$, then the above equation shows us that ω can only have simple zeros, and so $m\left(r, 1/\omega\right) = S(r, f)$. Using this and (4.19),

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{1}{\omega}\right) + m\left(r, \frac{\omega}{f}\right) = S(r, f),$$

and so, by the First Fundamental Theorem,

$$T(r, f) \leq N\left(r, \frac{1}{f}\right) + S(r, f).$$

Again using $1/f = (1/\omega)(\omega/f)$, and noting that each $f^{(s)}/f$ has poles of order at most s, it becomes clear that

$$T(r, f) \leq N\left(r, \frac{1}{f}\right) + S(r, f) \leq N^{(t)}\left(r, \frac{1}{f}\right) + S(r, f), \tag{4.26}$$

which is stronger than (4.15).

QED

Proof of Theorem 4.3.1 - Part III

We can now assume that H is meromorphic (since otherwise (4.15) holds by the previous lemma). We integrate (4.19) using Lemma 4.3.5, obtaining a polynomial $P(H)$ of degree at most $t-1$, such that

$$f = \frac{H^{t+1} + HP(H) - (-1)^t}{t!H(H')^t} = \frac{(H-c_1)\dots(H-c_{t+1})}{t!H(H')^t}$$

for constant roots c_s whose product has modulus 1. If the roots are all identical, then f has the form of (4.17), so assume without loss of generality that $c_1 \neq c_2$:

$$\frac{1}{t!f} = \frac{HH'}{(H-c_1)(H-c_2)} \prod_{s=3}^{t+1} \left(\frac{H'}{H-c_s} \right),$$

which in turn gives

$$\frac{1}{f} = \left(\frac{v_1 H'}{H-c_1} + \frac{v_2 H'}{H-c_2} \right) \prod_{s=3}^{t+1} \left(\frac{H'}{H-c_s} \right) \tag{4.27}$$

for some constants v_s. But this combined with (4.19) shows that

$$m\left(r, \frac{1}{f}\right) = S(r,H) = S(r,\psi) = S(r,f).$$

But (4.27) also tells us that f cannot have a zero of multiplicity more than t by the definition of H, so we obtain (4.26) once more.

We finally suppose that $a_{t-1} \equiv 0$, and also assume that (4.16) holds, and that f is as in (4.17). Then H is a linear function since

$$\frac{H''}{H'} = 0,$$

and so f is rational with $f(\infty) = \infty$. Then by the First Fundamental Theorem,

$$T(r,f) = N\left(r, \frac{1}{f}\right) + O(1)$$

for large r. Since f has only one zero, of multiplicity $t+1$, (4.15) holds.

QED

This result is a more general result than (4.13), and the paper's authors go on to give various examples of normal families of functions which satisfy the conditions of the given formula, with a view to showing that the constants in the result are not sharp. Also note that in case (ii), it is possible to have functions where $f \neq 0$ and $\psi \neq 1$.

Using this result, we may prove a new version of Hayman's Alternative for a specific case.

Corollary 4.3.7

Suppose that f is such that (i) *holds in Theorem 4.3.1, and that $a_0' \equiv 0$, such that*

$$\psi = f^{(t)} + \sum_{s=1}^{t-1} a_s f^{(s)} + \mu f$$

for some finite complex constant μ. Then either f takes every finite value infinitely often, or ψ takes every finite non-zero value infinitely often.

Proof:

The proof follows much like that of Corollary 4.2.6, in that we replace f with $F = (f - \omega_1)/\omega_2$, for some complex values ω_j with $\omega_2 \neq 0$, and then apply the inequality (4.15). However, we note the following:

$$L_t(F) = \frac{1}{\omega_2} L_t(f) - a_0 \frac{\omega_1}{\omega_2}.$$

To avoid complications, we let a_0 be some constant μ, and change the $N^{(p)}(r, 1/f)$s to plain $N(r, 1/f)$s, as this does not affect our required result, thus obtaining the inequality

$$T(r,f) \leq \left(2 + \frac{1}{q}\right) N(r,f) + \left(2 + \frac{2}{q}\right) \overline{N}\left(r, \frac{1}{\psi - \mu\omega_2 - \omega_2}\right) + S(r,f),$$

from which the result follows by the same logic as before. We note that ψ takes the value 0 infinitely often here if $\mu \neq 0$, as we can set $\omega_2 = -\mu\omega_1$.

QED

Thus, we have proved a more general version of Hayman's Alternative for the case that (i) holds in Theorem 4.3.1.

5 Zeros of certain differential polynomials

A large part of research in the field of Nevanlinna Theory focusses on its applications to complex differential equations. Ilpo Laine's text on differential equations [7] uses Nevanlinna Theory throughout the book, albeit in small doses. In this section I will present the results of some papers which reference [3], one of the most influential papers on the subject of Nevanlinna Theory. We will first take a brief look at that paper.

5.1 A paper by Bergweiler and Eremenko

Bergweiler and Eremenko's paper was published in 1995, and is interesting in a major way because it involves iterative complex analysis - something few papers on Nevanlinna Theory use. In particular, they used it to prove several conjectures by Hayman, including that $f'f$ must take every finite value infinitely often - for more on this, see section 5.2. We will look briefly at their methods.

Definition 5.1.1

Let f *be a meromorphic function, and* f^{-1} *its inverse. Further let* $a \in \mathbb{C}^*$*, and for every* $r > 0$*, choose a component* $U(r)$ *of the preimage* $f^{-1}(B(a,r))$ *in such a way that* $r_1 < r_2$ *implies* $U(r_1) \subset U(r_2)$*. Then two possibilities can occur:*

(i) $\cap_{r>0} U(r) = \{z_0\}$ *for some* $z_0 \in \mathbb{C}$*. If* $a \in \mathbb{C}$ *and* $f'(z_0) \neq 0$*, or if* $a = \infty$ *and* z_0 *is a simple pole of* f*, then* z_0 *is called an* ordinary point*. Otherwise, if* $a \in \mathbb{C}$ *and* $f'(z_0) = 0$*, or if* z_0 *is a multiple pole of* f*, then* z_0 *is called a* critical point *and* a *is called a* critical value*. An example of this is* $f = z^2$ *for* f^{-1} *near the origin.*

(ii) $\cap_{r>0} U(r) = \emptyset$*. We then say that our choice of* $U(r)$ *defines a* singularity *of* f^{-1}*. For every* $r > 0$*, the region* $U(r)$ *is called a* neighbourhood of the singularity

U. An example of this is $f = e^z$ for f^{-1} near the origin.

If U is some singularity then a is called its asymptotic value*, which means that there exists some curve $\Gamma \in \mathbb{C}$, tending to ∞, such that $f(z) \to a$ as $z \to \infty$. Such a Γ is called an* asymptotic curve.

It is worthwhile noting that it is possible to have more than one singularity, as well as critical and ordinary points, over the same point a, as we see in the following example.

Example 5.1.2

Let $f = z^2 e^{z^2}$, and let us look at the preimage of $f^{-1}(B(0,r))$. Clearly, $f(0) = f'(0) = 0$, and so the origin is a critical point. However, along the line $Re(z) = 0$, as $|z| \to \infty$, $f \to 0$, and so the origin is also a singularity.

We will now split singularities into two classes.

Definition 5.1.3

A singularity U over a is called direct *if there exists some $r > 0$ such that $f(z) \neq a$ for all $z \in U(r)$. For example, $f = e^z$ with $a = 0$. A singularity is called* indirect *if it is not direct, i.e. for every $r > 0$, there exists a $z \in U(r)$ such that $f(z) = a$. It is clear that f must take the value a infinitely often in $U(r)$, since if not, we can find some a-point that is closest to the point of singularity, say at range δ, and then we would find that there were no a-points in $U(\delta/2)$. An example of an indirect singularity is the inverse function to $f = \sin z/z$ at $a = 0$.*

The paper then goes on to prove several interesting results, which I present here without proof.

Proposition 5.1.4

Let f be a meromorphic function of finite order. Then every indirect singularity of f^{-1} over $a \in \mathbb{C}^*$ is a limit point of critical values $f(z_s) \neq a$.

This result is useful in the iteration of rational and transcendental meromorphic functions of finite order, and its corollary was used to prove Hayman's Conjecture, which makes up the first part of section 5.2.

Proposition 5.1.5

Let f be a meromorphic function of finite order. If f has infinitely many multiple zeros, then f' assumes every finite non-zero value infinitely often.

Bergweiler and Eremenko prove this result using iterative methods which are too involved to go into in this text. This proposition will be important in the next section, and generally provides a new version of Hayman's Alternative, replacing the requirement that f never take the value 0.

5.2 On the zeros of $(f^n)^{(t)}$

We turn now to the t^{th} derivative of f^n, for some positive integers n, t. The study of this sprang from Hayman's conjecture that, if f is a transcendental meromorphic function, then $ff' = \left((f/\sqrt{2})^2\right)^{(1)}$ takes every finite non-zero value infinitely often. This was proven in the paper by Bergweiler and Eremenko (Proposition 5.2.1). We will further look at variants on this function.

Proposition 5.2.1

Let f be transcendental and meromorphic in the plane, and let n and t be non-negative integers such that $n > t$. Then $(f^n)^{(t)}$ takes every finite non-zero value infinitely often.

The proof of this proposition uses normal families, as well as spherical derivatives and several lemmas by other authors. We now prove some corollaries.

Corollary 5.2.2

Let f be transcendental and meromorphic in the plane, and m a positive integer. Then both $f^m f'$ and $f^{(m)} f^{(m+1)}$ take every finite non-zero value infinitely often.

Proof:

We use Proposition 5.2.1 in both cases. For $f^m f'$, we let $t = 1$ and $n = m+1$, noting that the case $m = 1$ proves Hayman's conjecture. For the second case, we set $t = 1$, $n = 2$ and replace f by $f^{(m)}$.

QED

Corollary 5.2.3

Let f be a transcendental meromorphic function of finite order. Then $f' + f^m$ has infinitely many zeros for every integer $m \geq 3$.

A corresponding result to this was already known for $m \geq 4$, thanks to Mues and Hayman. We will show how Proposition 5.2.1 allows us to prove this more general case.

Proof:

First, let $g = 1/f$ for some function f. Then by Proposition 5.2.1, $(g^{m-1})'$ takes every finite non-zero value infinitely often. Let us say it takes the value j. Then for finite f

$$\left(g^{m-1}\right)' = \left(f^{1-m}\right)' = (1-m)f^{-m}f' = j,$$

and so

$$jf^m + (m-1)f' = 0.$$

We now set $j = m - 1$, and the result follows.

QED

We will now look at a paper by Pai Yang [13], where he deals with the special case of $n = 2$. The author does however note the following result by Yuefei Wang.

Proposition 5.2.4

Let f be transcendental and meromorphic in the plane, and let $n \geq 3$ and $t \geq 0$ be two integers. Then $(f^n)^{(t)}$ assumes every finite non-zero value infinitely often.

We also note that the case of $n = 1$, $t \geq 1$ is covered by Hayman's Alternative (Corollary 4.2.6), assuming the required conditions are satisfied. Thus we are left with only the case $n = 2$, for which Yang supplies the following result.

Proposition 5.2.5

Let f be a transcendental and meromorphic in the plane, and have only zeros of multiplicity at least $\left[\frac{t}{2}\right] + 1$. Then $(f^2)^{(t)}$ assumes every finite non-zero value infinitely often.

The proof of this theorem is rather involved for this text, but we shall prove it in a special case.

Proof of Proposition 5.2.5 for f of finite order

Suppose first that f has only finitely many zeros, then so does f^2, and our conclusion follows from Theorem 4.2.5. Now, let f have infinitely many zeros, and in particular let it have a zero of multiplicity j at some z_0. Then z_0 is a zero of $(f^2)^{(t-1)}$ with multiplicity at least $m = 2j-(t-1)$. Substituting in $j = \left[\frac{t}{2}\right]+1$, we see that $m \geq 2$, and so $(f^2)^{(t-1)}$ has infinitely many multiple zeros. The result now follows by Proposition 5.1.5.

QED

5.3 Another look at Hayman's Alternative

We now turn to a paper by Nevo, Pang and Zalcman, entitled "Picard-Hayman Behavior of Derivatives of Meromorphic Functions with Multiple Zeros" [10]. The authors note that it has been shown that in Corollary 4.2.7, it is possible to replace the requirement that $f \neq 0$ by the assumption that all zeros of f have sufficiently high multiplicity.

Proposition 5.3.1

Let f be transcendental and meromorphic in the plane, with zeros all of multiplicity at least 3. Then f' assumes each finite non-zero value infinitely often.

Proposition 5.3.2

Let f be meromorphic in the plane, with zeros all of multiplicity at least 3. Then if $f'(z) \neq \mu$ for all $z \in \mathbb{C}$ and some finite constant μ, f is constant.

This is the best possible result for 5.3.2, as the following example shows.

Example 5.3.3

Let $a, b \in \mathbb{C}$, and $a \neq b$. Then

$$f = \frac{(z-a)^2}{z-b} = z + (b-2a) + \frac{(a-b)^2}{z-b}$$

vanishes only at $z = a$, where it has a double zero. It is also clear that f' omits the value 1, and so Proposition 5.3.2 fails when 3 is replaced by 2.

Theorem 5.3.4

Let f be a transcendental function of finite order, meromorphic in the plane, with all zeros multiple. Then f' assumes every finite non-zero value infinitely often.

This immediately follows from Proposition 5.1.5 and Corollary 4.2.6. While Proposition 5.1.5 does not in general hold for functions of infinite order, the authors nonetheless proved that this result could be extended to functions of infinite order.

5.4 A further question on ff'

In this section we will look at a paper by Kit-Wing Yu [14] where he explores the formula $ff' - a$ where $T(r, a) = S(r, f)$. We saw earlier that if a is a finite constant then there are infinitely many solutions (Corollary 5.2.2). Yu bases his work on the following results by Q. D. Zhang.

Proposition 5.4.1

Let f be transcendental and meromorphic in the plane, and let a be a non-vanishing small function. Then

$$T(r,f) < \frac{9}{2}\overline{N}(r,f) + \frac{9}{2}\overline{N}(r,a,ff') + S(r,f) \tag{5.1}$$

as $r \to \infty$.

Corollary 5.4.2

Let f be transcendental and meromorphic in the plane, let a be a small function which is not identically zero, and let $\Theta(c,f)$ be as in Definition 4.2.1. Then if $\Theta(\infty,f) \geq 7/9$, $ff' - a$ has infinitely many zeros.

Proof:

We begin with (5.1), dividing through by $T(r,f)$ and adding $7/2$ to both sides

$$\frac{9}{2} - \frac{\overline{N}(r,f)}{T(r,f)} < \frac{7}{2} + \frac{9}{2}\frac{\overline{N}(r,a,ff')}{T(r,f)}.$$

We note that $N(r,\infty,f) \equiv N(r,f)$, and thus, taking the lim sup as $r \to \infty$,

$$\Theta(\infty,f) - \frac{7}{9} < \frac{\overline{N}(r,a,ff')}{T(r,f)}.$$

Thus, if $ff' - a$ has only finitely many zeros, the right hand side will tend to 0, and so $\Theta(\infty,f) < 7/9$.

QED

Hayman proved in 1959 if f has finite order and a is a polynomial that $ff' - a$ has infinitely many zeros. We will prove a more general case, where a is any small function which is not identically zero, and $\Theta(\infty,f) < 7/9$, since the case where $\Theta(\infty,f) \geq 7/9$ is proved by the above corollary. We first need a lemma.

Lemma 5.4.3

Let f ba a non-constant meromorphic function and $\phi = 1/a$ be a non-vanishing small meromorphic function with some positive integer t such that $\phi f^{(t)} \not\equiv \mu$ where μ is a finite constant, and finally let b and c be any two distinct, finite constants. Then

$$\begin{aligned} T(r,f) < N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\phi f^{(t)} - b}\right) + N\left(r,\frac{1}{\phi f^{(t)} - c}\right) - N(r,f) \\ -N\left(r,\frac{1}{(\phi f^{(t)})'}\right) + S(r,f) \qquad (5.2) \end{aligned}$$

as $r \to \infty$.

Proof:

We start with

$$\begin{aligned} m\left(r,\frac{1}{\phi f}\right) &\leq m\left(r,\frac{1}{\phi f^{(t)}}\right) + m\left(r,\frac{f^{(t)}}{f}\right) \\ &= m\left(r,\frac{1}{\phi f^{(t)}}\right) + S(r,f), \end{aligned}$$

where the second line is deduced using Theorem 4.1.1. Now, using this together with the following two equations

$$\begin{aligned} m\left(r,\frac{1}{\phi f}\right) &= T(r,\phi f) - N\left(r,\frac{1}{\phi f}\right) + O(1) \\ m\left(r,\frac{1}{\phi f^{(t)}}\right) &= T(r,\phi f^{(k)}) - N\left(r,\frac{1}{\phi f^{(t)}}\right) + O(1), \end{aligned}$$

yields

$$T(r,\phi f) \leq N\left(r,\frac{1}{\phi f}\right) + T(r,\phi f^{(t)}) - N\left(r,\frac{1}{\phi f^{(t)}}\right) + S(r,f). \qquad (5.3)$$

We now use the Second Fundamental Theorem, specifically (3.5), using $N_1(r,f)$ as defined in Definition 3.2.1, to give

$$T(r,\phi f^{(t)}) \le N\left(r,\frac{1}{\phi f^{(t)}}\right) + N\left(r,\frac{1}{\phi f^{(t)}-b}\right) + N\left(r,\frac{1}{\phi f^{(t)}-c}\right)$$
$$-N_1\left(r,\phi f^{(t)}\right) + S\left(r,\phi f^{(t)}\right). \tag{5.4}$$

as $r \to \infty$. Now suppose at some point f has a pole of order $p \ge 1$. Then that pole contributes $t+p-1 \ge p$ to the pole-counting terms of $N_1\left(r,\phi f^{(t)}\right)$, and so

$$N_1\left(r,\phi f^{(t)}\right) \ge N(r,f) + N\left(r,\frac{1}{(\phi f^{(t)})'}\right) + S(r,f).$$

This, combined with (5.3), (5.4), noting that $S\left(r,f^{(t)}\right) = S(r,f)$, and remembering that $T(r,\phi) = S(r,f)$ yields the desired result.

QED

We are now in a position to prove Yu's main theorem.

Theorem 5.4.4

Let f be a transcendental meromorphic function and a be a small meromorphic function which does not vanish identically. Then at least one of $ff'-a$ and $ff'+a$ has infinitely many zeros.

Proof:

Let $\phi = 1/a$, $F = f^2/2$, $k = 1$, $b = 1$ and $c = -1$. Then, applying Lemma 5.4.3 to F,

$$2T(r,f) < 2N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\phi ff'+1}\right) + N\left(r,\frac{1}{\phi ff'-1}\right) - 2N(r,f)$$
$$-N\left(r,\frac{1}{(\phi ff')'}\right) + S(r,f).$$

By our assumption that $\Theta(\infty,f) < 7/9$, we have that as $r \to \infty$,

$$\frac{N\left(r,\frac{1}{\phi ff'+1}\right) + N\left(r,\frac{1}{\phi ff'-1}\right)}{2T(r,f)} > 0,$$

from which the result follows.

QED

There are several variants upon these formulae which are studied, in particular $ff'' - a(f')^2$ was studied by Bergweiler, among others. However, the results presented here give a good idea of the research in complex analysis which uses Nevanlinna Theory and Hayman's results.

6 Conclusion

In this text, we have looked at the elementary basis of Nevanlinna Theory, the more complex elements, and then seen how various prominent authors have applied the theory in their own work. Of course, it is not only "classical" complex analysis that uses Nevanlinna Theory - authors such as Rippon and Stallard have used the theory in their investigations of complex dynamics.

One thing which the reader cannot have failed to notice is that all the work we have done involves a single complex variable. Unfortunately, Nevanlinna Theory requires considerable modifications to work in higher dimensions, as Hartog's Lemma states that in such dimensions, an isolated singularity is a removable singularity, and by its definition, $N(r, f)$ can only count isolated singularities. Other methods do exist to analyse these functions, but they are beyond the scope of this text.

In conclusion, we have seen how Nevanlinna Theory allows us to form startling conclusions about functions that we could not analyse using classical methods. We have seen a generalisation of Picard's Theorem, investigated a version of Fermat's Last Theorem for functions, proved Hayman's Alternative and looked at some applications to differential equations. From all of this we have seen how powerful a tool Nevanlinna Theory is, and I would like to hope that the reader derived as much enjoyment from the subject as I did.

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