Nevanlinna Theory

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I have read and understood the School and University guidelines on plagiarism.

I confirm that this work is my own, apart from the acknowledged references.

Aims

My dissertation is on the topic of Nevanlinna Theory, a powerful tool from complex analysis. My aim is to provide an overview of the basic theorems and results of the subject, and then investigate where it has been used in research.

Work so far

Nevanlinna Theory is primarily concerned with the study of meromorphic functions - functions which are complex differentiable at all points except an isolated set of poles. Such functions are usually written in the form $f(z) = \dfrac{A(z)}{B(z)}$ where the poles of f are the zeroes of B. Such functions cannot be analysed using standard tools such as the maximum modulus function due to their poles. Instead, we let $n(r, f)$ be the number of poles of f in the disc $|z| \leq r$, and hence define:

$$N(r, f) = \int_0^r (n(t, f) - n(0, f)) \frac{dt}{t} + n(0, f) \log r$$

The N function counts the number of poles, with regards to multiplicity - a pole of multiplicity n adds n to $n(r, f)$. We further define $\overline{N}(r, f)$ as the number of poles of a function without regards to multiplicity - each pole adds 1 to $\overline{n}(r, f)$. We further set

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| re^{i\phi} \right| d\phi,$$

$$T(r, f) = N(r, f) + m(r, f),$$

where $\log^+ x$ is defined as $\max\{0, \log x\}$. The function $T(r, f)$ is known as the *Nevanlinna Characteristic Function*.

From this, through work done by Cartan, we come to the *Second Fundamental Theorem* (2FT).

$$(n-2)T(r,f) \leq \sum_{j=1}^{n} \overline{N}\left(r, \frac{1}{f-a_j}\right) + O(\log^+(rT(r,f))).$$

From this, Picard's Theorem is a corollary.

This term, I have also done work on the following:

I have shown that $\dfrac{(f')^2}{(f-a_1)(f-a_2)}$ is entire, and indeed is a polynomial, for any f and complex constants a_1 and a_2 where the zeroes of $f-a_i$ have multiplicity ≥ 2. This was proved by looking at the multiplicity of zeroes of $f-a_i$: if $f-a_i$ has a zero of multiplicity n at a point z, then f' must have a zero of multiplicity $n-1$ at z.

Work on the Weierstrass $\wp$-function. This is a function with two periods, whose poles form a lattice on $\mathbb{C}$.

Fermat's Last Theorem for functions: $f^n + g^n = 1$. We know that $\sin^2 z + \cos^2 z = 1$ is a standard identity, and thus there are functions where the Fermat equation holds for $n = 2$, but it can be shown by use of the 2FT that there are no entire functions satisfying this equation for $n \geq 3$, and no meromorphic functions for $n \geq 4$. I have further generalised this to $af^n + bg^n = 1$ where a, b are rational functions. I also checked Fred Gross's example of meromorphic functions where the equation holds for $n = 3$.

I have also looked into the proofs by Hayman which show that all derivatives of transcendental functions must take every finite value, with at most one exception, infinitely often.

Work Plan

Throughout the second semester, I plan to investigate recent literature dealing with and related to problems that I have looked at already. I will also look into functions taking the same values at the same points: the Nevanlinna Five-Value Theorem. I will use MathSciNet, as well as various reference texts, for this work.

References

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[5] Einar Hille, *Ordinary Differential Equations in the Complex Domain*, (John Wiley and Sons, 1976).

[6] J. V. Armitage and W. F. Eberlein *Elliptic Functions* (Cambridge University Press, 2006).